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MATHEMATICS

FOR COLLEGIATE STUDENTS OF

AGRICULTURE AND GENERAL SCIENCE

REVISED EDITION

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MATHEMATICS

FOR COLLEGIATE STUDENTS OF

AGRICULTURE AND GENERAL SCIENCE

REVISED EDITION

BY

ALFRED MONROE KENYON

PROFESSOR OF MATHEMATICS IN PURDUE UNIVERSITY

AND

WILLIAM VERNON LOVITT

ASSOCIATE PROFESSOR OF MATHEMATICS IN COLORADO COLLEGE

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PREFACE

This book is designed as a text in freshman mathematics for students specializing in agriculture, biology, chemistry, and physics, in colleges and in technical schools.

The selection of topics has been determined by the definite needs of these students. An attempt has been made to treat these topics and to select material for illustration so as to put in evidence their close and practical relations with everyday life, both in and out of college. It is certain that the interest of the student can be aroused and sustained in this way. We believe also that he can be trained to understand and to solve those mathematical problems which will confront him in the subsequent years of his college work and in after-life, without losing anything in orderly arrangement or in clear and accurate logical thinking.

Reference to the table of contents will indicate the scope and proportions of the material presented and something of the means employed in relating the material to the vital interests of the student and of correlating it to his experience and his intellectual attainments. Many of the chapter subjects and paragraph headings are traditional. Nothing has been introduced merely for novelty. Since this course is to constitute the entire mathematical equipment of some students, some chapters have been inserted which have seldom been available to freshmen; for example, the chapters on annuities, averages, and correlation, and the exposition of Mendel's law in the chapter on the binomial expansion.

Particular attention has been given to the illustrative examples and figures, and to the grading of the problems in the lists. The exercises constitute about one fifth of the text and contain

a wealth of material. They include much data taken from agricultural and other experiments, carefully selected to stimulate thinking and to show the application of general principles to problems which actually arise in real life, and in the solution of which ordinary men and women are vitally interested.

The book is intended for a course of three hours a week for one year, but it can be shortened to a half-year course. The chapters on statics, small errors, land surveying, annuities, compound interest law, and as many as is desired at the end, can be omitted without breaking the continuity of the course.

The first two chapters are more than a mere review. This matter is so presented as to give the student a new point of view. The treatment will show the significance and importance of certain fundamental relations among the concepts and processes of arithmetic and algebra which the student may have used somewhat mechanically in secondary school work. Well prepared students can read these chapters rather rapidly, however.

The four place mathematical tables printed at the end of the text have been selected and arranged for practical use as the result of long experience and actual use in computing, and are adapted to the requirements of the examples and exercises in the book.

The first edition of this book contained problems, formulas and other matter taken from a large number of sources. Those passages that were directly from other books have now been entirely rewritten; but the book remains indebted to a number of others, notably Skinner, *Mathematical Theory of Investment*, and Davenport, *Principles of Breeding*. Other references occur throughout the text.

A. M. Kenyon,
W. V. Lovitt.

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MATHEMATICS

CHAPTER I

INTRODUCTION

1. Uses of Mathematics. The applications of mathematics are chiefly to determine the magnitude of some quantity such as length, angle, area, volume, mass, weight, value, speed, etc., from its relations to other quantities whose magnitudes are known, or to determine what magnitude of some such quantity will be required in order to have certain prescribed relations to other known quantities.

2. Measurement. To measure a quantity is to find its ratio to a conveniently chosen unit of the same kind. This number is called the numerical measure of the quantity measured.

The expression of every measured quantity consists of two components: a *number* (*the numerical measure*), and a *name* (that of the *unit* employed). For example, we write: 10 inches, 27 acres, 231 cubic inches, 16 ounces, 22 feet per second.

3. Arithmetic and Algebra. In arithmetic we study the rules of reckoning with positive rational numbers. In algebra negative, irrational, and imaginary numbers are introduced, letters are used to represent classes of numbers, and the rules of reckoning are extended and generalized. Algebra differs from arithmetic also in making use of equations for the solution of problems requiring the discovery of numbers which shall satisfy certain prescribed conditions.

4. Positive Numbers. The *natural numbers* 1, 2, 3, 4, etc., are the foundation on which the whole structure of mathematics is built. They are also called *whole numbers*, or *positive integers*. Together with the *fractions*, of which 1/2, 5/3, .9, 2.31, are examples, they form the class of *positive rational numbers*.

Every positive rational number can be expressed as a fraction whose numerator and denominator are whole numbers.

Two quantities of the same kind are said to be *commensurable* when there is a unit in terms of which each has for numerical measure a whole number. Consequently, their ratio is a rational number. If two quantities are not commensurable, they are said to be *incommensurable*.

The ratio of two quantities which are incommensurable, such as the side and the diagonal of a square, or the diameter and the circumference of a circle, is an *irrational number*.

No irrational number can be expressed as a fraction whose numerator and denominator are whole numbers. However, it is always possible to find two rational numbers, one less and the other greater than a given irrational number, whose difference is as small as we please. For example,

$$3.162277 < \sqrt{10} < 3.162278$$

and the difference between the first and the last of these numbers is only .000001. Two such rational numbers whose difference is still less can easily be found. In all practical applications, one of these rational numbers is used as an approximation for the irrational number. Thus, we may find the length of the circumference of a circle approximately by multiplying its diameter by $3\frac{1}{7}$. If a closer approximation is needed, the value 3.1416 is often used.

The (positive) rational and the (positive) irrational numbers make up the class of (positive) *real numbers*.

5. Negative Numbers. Zero. To every positive real number r, there corresponds a negative real number $-r$, called negative r. The negatives of the natural numbers are called negative integers. The real number zero separates the negative numbers from the positive numbers. It is neither positive nor negative and corresponds to itself.

The negatives of negative numbers are the corresponding positive numbers; thus, $-(-2) = 2$.

6. The Four Fundamental Operations. The direct operations of ***addition*** and ***multiplication*** of real numbers are so defined that they are always possible, and so that the result in each case is a unique real number. These operations are subject to the rules of signs and to the following fundamental laws of algebra.

I. *The commutative law:*

$$a + b = b + a, \qquad ab = ba.$$

II. *The associative law:*

$$(a + b) + c = a + (b + c), \qquad (ab)c = a(bc).$$

III. *The distributive law:*

$$a(b + c) = ab + ac.$$

The indirect operations of ***subtraction*** and ***division*** of real numbers are always possible, division by zero excepted,* and the result is a unique real number.

7. Involution and Evolution. *Involution*, or raising to powers, is always possible, and the result is unique when the base is any real number provided the exponent is a positive integer.

* Division by zero is excluded because, in general, it is impossible, and when possible it is trivial. Thus there is no real number which will satisfy the equation $0 \cdot x = a \neq 0$, and every real number satisfies the equation $0 \cdot x = 0$.

Evolution, or extraction of roots,* is not always possible. Even when possible, it is not always unique. In particular, the square of every real number is a positive real number. Hence no negative number can have a real square root. On the other hand, every positive real number, a, has two real square roots: a positive one, which is denoted by the symbol $\sqrt{a}$; and a negative one, which is denoted by $-\sqrt{a}$. In fact, every positive real number has exactly two real nth roots of every even index n, denoted by $\sqrt[n]{a}$ and $-\sqrt[n]{a}$, respectively. Every real number, r, has a unique real nth root of every odd index n, denoted by $\sqrt[n]{r}$; it is positive when r is positive, and negative when r is negative.

8. Rational Exponents. Involution is extended to fractional exponents as follows. If a is a positive real number, and if m and n are natural numbers, we define $a^{m/n}$ by the equation

$$a^{m/n} = \sqrt[n]{a^m}.$$

For example,

$$8^{2/3} = \sqrt[3]{64} = 4.$$

In particular, $a^{1/n} = \sqrt[n]{a}$ denotes the unique real positive nth root of a.

If r is a positive rational number, $(-a)^r$ is defined only when r, expressed as a fraction m/n, in its lowest terms, has an odd denominator and in this case,

$$(-a)^r = (-1)^m a^r.$$

For example, $(-32)^{.6} = (-32)^{3/5} = (-1)^3(32)^{3/5} = -8$, and $(-32)^{.8} = (-32)^{4/5} = (-1)^4(32)^{4/5} = 16$.

In particular, $(-a)^{1/n} = -a^{1/n} = -\sqrt[n]{a}$, if n is odd.

* The index of a root is always a positive integer.

9. Negative and Zero Exponents. By definition, we write

$$a^{-b} = \frac{1}{a^b} \qquad \text{and} \qquad a^0 = 1,$$

provided $a \neq 0$. Thus a^{-b} is defined for the same real values of a and b as is a^b and the two are reciprocals.* For example,

$$8^{-2/3} = \frac{1}{8^{2/3}} = \frac{1}{4}, \qquad \left(\frac{1}{9}\right)^{-5/2} = \frac{1}{\left(\frac{1}{9}\right)^{5/2}} = 243.$$

A consequence of this definition is the rule: *A factor may be moved from the numerator to the denominator of a fraction, or vice versa, on changing the sign of its exponent.* For example,

$$\frac{a^2bc^{-3}}{2de^{-1}} = \frac{2^{-1}d^{-1}e}{a^{-2}b^{-1}c^3} = \frac{a^2be}{2c^3d},$$

$$\frac{1}{\sqrt{a^2 - x^2}} = (a^2 - x^2)^{-1/2},$$

$$x^{-2} + 2x^{-1}y^{-1} + y^{-2} = \frac{1}{x^2} + \frac{2}{xy} + \frac{1}{y^2}.$$

EXERCISES

1. Verify the fundamental laws of algebra by making use of the three numbers $\frac{2}{3}$, $-5\frac{1}{4}$, $\frac{3}{8}$.

2. How many real square roots has 24.5? -4.5?

3. How many real cube roots has $6\frac{2}{5}$? $-12\frac{1}{2}$?

4. Find the numerical value of each of the following expressions, exactly when rational, correct to three decimal places when irrational.

(*a*) $9^{3/2}$.	(*e*) $(32)^{3/5}$.	(*i*) $(-0.027)^{1/3}$.
(*b*) $(\frac{8}{27})^{2/3}$.	(*f*) $(-32)^{2/5}$.	(*j*) $(\frac{1}{2}\frac{3}{7})^{6/4}$.
(*c*) $2^{4/3}$.	(*g*) $(\frac{3}{5})^{1/3}$.	(*k*) $(\frac{2}{5})^{1/2}$.
(*d*) $(-2)^{4/3}$.	(*h*) $(-0.375)^{2/3}$.	(*l*) $(-\frac{2}{3}\frac{4}{1})^{5/3}$.

*Two numbers are reciprocals when their product is $+1$. Every real number has a reciprocal except 0, which has none.

5. Write each of the following expressions without radical signs.

(*a*) $\sqrt[3]{32}$. (*b*) $\sqrt{128}$. (*c*) $\sqrt[5]{125}$. (*d*) $\frac{1}{3}\sqrt[3]{a^2b^4c^5}$.

(*e*) $6\sqrt{\frac{8x^3}{81y^5}}$. (*f*) $\frac{1}{2x}\sqrt[4]{\frac{c^5y}{a^2b^2}}$. (*g*) $3.2\sqrt{\frac{m+n}{2048}}$.

(*h*) $4\sqrt[3]{(a-b)^2}$. (*i*) $\sqrt[3]{4^2 \cdot 8^4}$. (*j*) $\sqrt[3]{(a^2-b^2)(a+b)^2}$.

6. Write each of the following expressions without negative exponents and simplify when possible.

(*a*) $(\frac{2}{3})^{-1}$. (*b*) $(\frac{5}{4^0})^{-1/3}$. (*c*) $\left(\frac{12.8}{5}\right)^{-3/2}$.

(*d*) $(27x^{-6}y^3z^{-12})^{-1/3}$. (*e*) $(a^{-2}+b^{-2})^{-1}$. (*f*) $(a^{-2}b^{-2})^{-1}$.

(*g*) $\dfrac{(x+y)^{-1}}{x^{-1}+y^{-1}}$. (*h*) $\dfrac{a^{-2}-b^{-2}}{a^{-1}-b^{-1}}$. (*i*) $\dfrac{x^0y^{-1}-xy^0}{x^{-1}y^0+x^0y}$.

(*j*) $\dfrac{3^0a^{-1}b^2-3^{-1}a^2b^0}{3^2a^0b^{-1}+3ab}$. (*k*) $\dfrac{x^{-1}y^{-2}z^{-3}+x^3y^2z}{x^{-3}y^{-2}z^{-1}+xy^2z^3}$.

10. Laws of Exponents. The following five laws are useful for the reduction of exponential and radical expressions to simpler forms. They are valid, (1) when the bases are any real numbers whatever, provided the exponents are integers or zero, and (2) when the exponents are any real numbers whatever, provided the bases are positive.

I. $a^b \cdot a^c = a^{b+c}$.

EXAMPLES. $3^2 \cdot 3^{-4} = 3^{-2}$. $(-\frac{2}{3})^5(-\frac{2}{3})^{-3} = (-\frac{2}{3})^2$.

$(\frac{2}{5})^{-1/3}(\frac{2}{5})^{1/2} = (\frac{2}{5})^{1/6}$. $8^{-1/3} \cdot 8^2 = 8^{5/3}$.

II. $a^cb^c = (ab)^c$.

EXAMPLES. $2^35^3 = 10^3$. $(-3)^{-2}(-5)^{-2} = (15)^{-2}$.

$(17)^{1/3}(\frac{5}{17})^{1/3} = 5^{1/3}$.

III. $(a^b)^c = a^{bc}$.

EXAMPLES. $(2^2)^2 = 2^6$. $[(-\frac{2}{3})^{-2}]^{-3} = (-\frac{2}{3})^6$.

$[(\frac{2}{5})^{1/3}]^6 = (\frac{2}{5})^2$.

IV. $$\frac{a^b}{a^c} = a^{b-c}.$$

EXAMPLES. $$\frac{3^2}{3^{-1}} = 3^3. \qquad \frac{(-\frac{5}{6})^{1/2}}{(-\frac{5}{6})^{-1/3}} = (-\tfrac{5}{6})^{5/6}.$$

V. $$\frac{a^c}{b^c} = \left(\frac{a}{b}\right)^c.$$

EXAMPLES. $$\frac{4^{-2/5}}{3^{-2/5}} = (\tfrac{4}{3})^{-2/5}. \qquad \frac{(\frac{2}{7})^{-2}}{(\frac{3}{7})^{-2}} = (\tfrac{2}{3})^{-2}.$$

These laws are readily proved when the exponents are positive integers. Thus, to prove law II, when the exponent is a positive integer n, we write

$$a^n \cdot b^n = \overset{(1)}{a} \cdot \overset{(2)}{a} \cdot \overset{(3)}{a} \cdots \overset{(n)}{a} \cdot \overset{(1)}{b} \cdot \overset{(2)}{b} \cdot \overset{(3)}{b} \cdots \overset{(n)}{b}$$
$$= \overset{(1)}{ab} \cdot \overset{(2)}{ab} \cdot \overset{(3)}{ab} \cdots \overset{(n)}{ab} = (ab)^n.$$

Similarly, each of the other laws can be proved when the exponents are positive integers. When the exponents are negative, we make use of the definition of § 9. If they are positive fractions we make use of the following lemma: *If a and b are real numbers of like sign, and if $a^n = b^n$, where n is a positive integer, then $a = b$.*

11. Binomial Theorem. By multiplying out, we find the following equalities:

$$(x + y)^2 = x^2 + 2xy + y^2,$$
$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3,$$
$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4,$$
$$(x + y)^5 = x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5.$$

By observing the coefficients and the exponents of x and of y in the various terms, we observe the law by which these results can be written down without the work of multiplying them out.

In the expansion of $(x + y)^n$ for $n = 2, 3, 4, 5$, we note the following facts:

(1) The number of terms is $n + 1$.

(2) The exponent of x in the first term is n and it decreases by 1 in each succeeding term; the exponent of y in the second term is 1 and it increases by 1 in each succeeding term.

(3) The first coefficient is 1; the second is n; the coefficient of any term after the second may be found from the preceding term *by multiplying the coefficient by the exponent of x and dividing by a number* 1 *greater than the exponent of y.*

These three statements constitute the ***binomial theorem,*** which will be proved in § 208, Chapter XVI, for all values of x and y no matter how large the positive integer n may be. The coefficients which appear in these expansions are called ***binomial coefficients.*** For example, the numbers

$$1, \quad 5, \quad 10, \quad 10, \quad 5, \quad 1$$

are the binomial coefficients for the fifth power. The binomial coefficients for the second, third, fourth, and fifth powers should be memorized.

EXERCISES

Use the laws of exponents to combine and simplify the following expressions.

1. $8^{-1/2} \cdot 8^{2/3} \cdot 8^{-1/5} \cdot 8^{2} \div 8^{3/4} \cdot 8^{1/12}$.

2. $3^{2/5} \cdot 4^{2/5} \cdot 5^{2/5} \div 15^{2/5} \cdot 8^{2/5}$.

3. $(3^{2} \cdot 3^{1/2} \cdot 5^{5/2})^{2} \div (7^{3} - 10^{2})$.

4. $(11 \cdot 3^{2} + 7^{4})^{1/2}$.

5. $(5^{4} - 3^{2} \cdot 2^{6})^{1/2}$.

6. $\dfrac{8^{3/4}}{8^{5/12}}$.

7. $\dfrac{12^{3/2}}{3^{3/2}}$.

8. $\dfrac{40^{2/3}}{5^{2/3}}$.

9. $\dfrac{\sqrt{48}}{\sqrt{3}}$.

10. $\dfrac{\sqrt{54}}{\sqrt[4]{36}}$.

11. $\dfrac{\sqrt[3]{12}}{\sqrt{6}}$.

12. $\sqrt{a^{3}b^{5}}\sqrt[3]{ab^{4}}\sqrt[6]{a^{7}b}$.

13. $\sqrt[3]{m^{4}n^{5}}\sqrt[3]{m^{2}n^{4}}$.

14. $\left(\dfrac{x^{-1/3}y^{5/6}}{z^{-3/4}}\right)^{-12}$.

15. $\left(\dfrac{x^{-2} - y^{-2}}{x^{-1} + y^{-1}}\right)^{-2}$.

Perform the indicated operations and simplify each of the following expressions when possible.

16. $\left(\frac{2ax^2}{3b^2y}\right)^3$.

17. $\left(\frac{2a^5x}{3a^2x^3}\right)^{-2}$.

18. $\left(\frac{a^{-2}x^2y^4}{b^{-2}x^4y^2}\right)^{-3/2}$.

19. $\left(\frac{a^{-1}b^{-2}c^{-3}}{x^0yz^2}\right)^{-1}$.

20. $\left(\frac{3a^2b^3}{5x^3y^2}\right)^{-3}$.

21. $\left(\frac{16x^4y^8}{a^{12}b^{12}}\right)^{-3/4}$.

22. $\left(\frac{x+2}{x^2+x-2}\right)^{-1}$.

23. $\left(\frac{a+b}{a-b}\right)^2 \div \left(\frac{a^3+b^3}{a^2-b^2}\right)$.

24. $63a^4x^5 \div 9a^3x^2 \div 3a^2x$.

25. $(a^0+b)(a+b^0)$.

26. $\frac{72a^3b^4x^2y^5}{18a^4b^3x^5y^2}$.

27. $\left(\frac{2a^0bx^2y^3}{5a^3b^2xy^0}\right)$.

28. $\frac{6a^3b^5+9a^2b^3+15a^4b^9}{3a^2b^2}$.

29. $\frac{x-y}{x+y} \times \frac{x^3+y^3}{x^2-y^2}$.

30. $\frac{2a^2+7ax+3x^2}{2a+x} \div \frac{3a^2+7ax+2x^2}{a+3x}$.

31. $\frac{x^2-7x}{x^2-5x+4} \times \frac{x^2-4x+3}{x^2-10x+21} \div \frac{x^2-5x}{x^2-9x+20}$.

32. $\frac{a^2+5a+4}{a^2-a-20} \div \frac{a+3}{a^2-2a-15} \times \frac{a^2+a-2}{a^2-a} \div \frac{a^2+3a+2}{a^2}$.

33. $\left(\frac{a-b}{b}+\frac{b}{a+b}\right) \div \left(\frac{b-a}{a}+\frac{a}{b+a}\right)$.

Multiply:

34. $a^{5/6}-a^{2/3}b^{1/3}+a^{1/6}b^{4/3}-b^{5/3}$ by $a^{1/6}+b^{1/3}$.

35. $a^{1/2}+2b^{1/2}-3c^{1/2}$ by $a^{1/2}-2b^{1/2}+3c^{1/2}$.

36. $x^{4/3}+2x+3x^{2/3}+2x^{1/3}+x^0$ by $x^{2/3}-2x^{1/3}+x^0$.

37. $\sqrt{x^3-x^2y-xy^2+y^3}$ by $\sqrt{x^3+3x^2y+3xy^2+y^3}$.

38. $a^{1/4}-b^{1/4}$ by $a^{3/4}+b^{3/4}$.

39. $x^{3/5}-y^{2/5}$ by $x^{2/5}+y^{3/5}$.

40. $(a^{1/2}b^{1/2}+c^{1/2})^2$ by $(a^{1/2}b^{1/2}-c^{1/2})^2$.

41. $(a^{-1/2}-3)^2$ by $(3a^{1/2}+1)^2$.

Divide:

42. $x^{5/2}+x^2-2x^{1/2}+1$ by $x+x^{1/2}-1$.

43. $x^3+27x-9x^{1/2}-10$ by $x-3x^{1/2}+5$.

44. $x-y-6x^{2/3}+12x^{1/3}-8$ by $x^{1/3}-y^{1/3}-2$.

45. $a^{5/2}-a^2b+a^{3\,2}c-ac+a^{1/2}b-1$ by $a^{1/2}-1$.

46. $a^2+8a^{1/4}+7$ by $a^{1/2}+2a^{1/4}+1$.

Reduce each of the following to its simplest form:

47. $\sqrt{12.25x^4y^6}$.
48. $\sqrt[3]{15.625a^6b^9}$.
49. $\sqrt[5]{343a^{10}b^{25}}$.
50. $\sqrt{3a^2b - 2a^2c}$.
51. $\sqrt{(a^3 + b^3)(a^2 + b^2 - ab)}$.
52. $\sqrt{x^4y^2 - 2x^3y^3 + x^2y^4}$.
53. $\sqrt[3]{\sqrt{1024}}$.
54. $\sqrt[3]{3\sqrt{3}}$.
55. $\sqrt{27a\sqrt[3]{27ab^4}}$.
56. $\sqrt[4]{\sqrt[3]{16a^2}}$.
57. $\sqrt[3]{1.35a^2\sqrt{6.25a^2}}$.
58. $\sqrt{32x\sqrt[3]{8x^9}}$.
59. $\sqrt{405} - 2\sqrt{605} + \sqrt{845}$.
60. $\sqrt[3]{192} - 2\sqrt[3]{375} + \sqrt[3]{648}$.
61. $\sqrt{72} - \sqrt{8} - \sqrt{50}$.
62. $\sqrt[3]{81} - 2\sqrt[3]{192} + \sqrt[3]{375}$.
63. $(\sqrt{153} - \sqrt{117} + \sqrt{52} - \sqrt{68})(\sqrt{51} + \sqrt{39})$.
64. $(\sqrt{12} + \sqrt{3} + \sqrt{5\tfrac{1}{3}})(\sqrt{12} - \sqrt{3} + \sqrt{5\tfrac{1}{3}})$.
65. $(2 + \sqrt{3} + \sqrt[3]{4})(2 - \sqrt{3} + \sqrt[3]{4})$.
66. $(3\sqrt{20} - 4\sqrt{5} + 5\sqrt{2} - 3\sqrt{8})(\sqrt{5} + \sqrt{0.5})$.
67. $\sqrt[3]{19 + 3\sqrt{2}} \cdot \sqrt[3]{19 - 3\sqrt{2}}$.
68. $\sqrt[5]{16 + \sqrt{13}} \cdot \sqrt[5]{16 - \sqrt{13}}$.

Rationalize the denominator of each of the following fractions:

69. $\dfrac{1 - \sqrt{2}}{3 - 2\sqrt{2}}$.
70. $\dfrac{1 - \sqrt{5}}{2 + \sqrt{5}}$.
71. $\dfrac{\sqrt{5} + \sqrt{2}}{2\sqrt{5} + 3\sqrt{2}}$.
72. $\dfrac{\sqrt{3} + \sqrt{2}}{\sqrt{3} - \sqrt{2}} + \dfrac{\sqrt{2} - \sqrt{3}}{\sqrt{2} + \sqrt{3}}$.
73. $\dfrac{2 + \sqrt{5}}{2 - \sqrt{5}} \times \dfrac{5 - \sqrt{2}}{5 + \sqrt{2}}$.
74. $\dfrac{\sqrt{2} - \sqrt{3}}{\sqrt{32} + \sqrt{48} - \sqrt{50} - \sqrt{75}}$.
75. $\dfrac{\sqrt{189} + 3\sqrt{20}}{\sqrt{84} - \sqrt{80}}$.

Expand each of the following expressions:

76. $(2p + 3q)^2$.
77. $(5c - 9d)^2$.
78. $(4m - 3n)^2$.
79. $(1 + x)^2$.
80. $(1 - x)^2$.
81. $(\frac{2}{3}a + \frac{3}{2}b)^2$.
82. $\left(1 + \dfrac{1}{x}\right)^2$.
83. $\left(1 - \dfrac{1}{x}\right)^2$.
84. $(\frac{3}{5}x - \frac{5}{2}y)^2$.
85. $(1 + x^2)^2$.
86. $(1 - x^2)^2$.
87. $(1 + \sqrt{x})^2$.
88. $(k^2 + 3)^2$.
89. $(2l^2 + 5)^2$.
90. $(a^2 + ab)^2$.
91. $(\sqrt{a} + \sqrt[3]{a})^2$.
92. $(a^{-1/2} + x^{1/2})^2$.
93. $(b^{1/3} - y^{1/2})^2$.
94. $(2x - 3y)^3$
95. $(a + \frac{1}{2}b)^3$.
96. $(\sqrt{m} + \sqrt[3]{m})^3$.
97. $(1 + x)^3$.
98. $(1 - x)^3$.
99. $(1 + x^2)^3$.
100. $[(a + b) + k]^2$.
101. $(x + y - a)^2$.
102. $(a^2 + ab + b^2)^2$.

CHAPTER II

REVIEW OF EQUATIONS*

12. Use of Equations. As indicated before, the chief advantage of algebra over arithmetic in solving problems lies in the method of attack. The algebraic method is to translate the problem into an equation and then to solve the equation by general methods.

13. Definition of an Equation. An *equation* is a statement of the equality of two expressions. Each of the expressions may contain letters and figures called knowns, representing numbers supposed to be given or known; letters called unknowns, representing numbers to be found; and symbols of operation and combination, such as $+$, $-$, etc.

As examples of equations in one unknown, we may write

(1) $$x + 13 = 2x - 7,$$

(2) $$x(5 - x) = 2(x + 1)(x^2 - x + 1),$$

(3) $$x(x + 2) = (x - 1)(x - 2) + 5x - 2,$$

(4) $$\frac{2x + 1}{x - 1} - \frac{2x - 1}{x + 1} = 4,$$

(5) $$7\sqrt{x - 6} + 6\sqrt{3x + 4} = 4x + 3.$$

As examples of equations in two unknowns, we may write

(6) $$x^2 - y^2 + 2y = 1,$$

(7) $$(2x - y)^2 - 5x^2 = 5y^2 - (x + 2y)^2.$$

Similarly, we may have equations in more than two unknowns.

*This chapter is intended for review work. Parts of it may be omitted at the discretion of the instructor, if it appears that the students do not need to review some of the topics.

The expression on the left of the equality sign is called the *left member*, or the *left side*, of the equation. The other is called the *right member*, or *right side*.

14. Substitution. It is often necessary to substitute for the unknowns in an expression such as one of the members of the above equations, certain definite numbers, called values of the unknowns. The result of such substitution is, in general, to reduce the expression to a single number.

Thus, if we put 10 for x in equation (1), the left side reduces to 23 and the right side to 13. If we put 20 for x, each member reduces to the same number, 33.

Again, if we put 1 for x and 1 for y in equation (6), the left side reduces to 2 and the right side to 1; but if we put 2 for x and -1 for y, each member reduces to 1.

15. Solution of an Equation. Any set of values of the unknowns which reduces each of the two members of an equation to the same number is said to ***satisfy*** the equation, and to be a ***solution*** of the equation. A solution of an equation in one unknown is also called a ***root*** of the equation.

The final test to determine whether a set of values of the unknowns in an equation is a solution or not, is to substitute these values for the unknowns and see whether the equation is satisfied or not.

For example, $x = 20$ is a solution of equation (1), § 13. The value $x = 10$ does not satisfy it. Again, $x = \frac{1}{2}$, $x = 1$, $x = -2$, are three solutions of (2). Every real number is a solution of (3). The value $x = 2$ is a solution of (4). The value $x = 15$ is a solution of (5). The values $x = 2$, $y = -1$ constitute a solution of (6). Every pair of real numbers constitutes a solution of (7).

16. Identities. An equation which is satisfied by all values of the unknowns (excepting those values if there are any for which either member is not defined) is called an ***identity***. An

equation which is not an identity is called a ***conditional equation,*** or when no ambiguity is likely to arise, simply an ***equation.***

EXAMPLES. Of the equations in § 13, (3) and (7) are identities, the others are conditional equations. Also,

$$\frac{1+\frac{1}{x}}{1-\frac{1}{x}} = \frac{x+1}{x-1}$$

is an identity; it is satisfied by all values of x, except $x = 1$ for which neither side is defined.

The distinction in point of view between identities and conditional equations is fundamental. To show that an equation is *not* an identity, we need only find a single set of values of the unknown quantities for which both sides are defined, and for which the equation is not true.

EXERCISES

1. Which of the numbers -3.5, -2, -1, 0, $\frac{1}{2}$, 2, satisfy the equation

$$\tfrac{1}{3}x + \frac{x+2}{3} = \frac{10}{2x+1}?$$

2. Which of the numbers $\frac{1}{10}\sqrt{7}$, $2+\sqrt{3}$, $\sqrt{14}$, $2-\sqrt{3}$, are solutions of the equation $x^2 + 1 = 4x$?

3. Which of the following pairs of numbers (0, 0), (1, 3), (4, 2), (0, 2), (1, -1), (3, -1), (4, 0), (3, 3), satisfy the equation

$$x^2 + y^2 = 4x + 2y?$$

Is this equation an identity?

4. Which of the following pairs of numbers (0, 1), (1, 1), (-1, 0), (2, 3), (-2, 1), (1, -1), (3, -2), are solutions of the equation

$$\frac{x+2y}{x+y} = 1 + \frac{y}{x}\left(1 - \frac{y}{x+y}\right)?$$

Is this equation an identity?

5. Which of the following equations are identities?

(*a*) $x(x^2 - y^2) = (x + y)(x^2 - xy)$.

(*b*) $x(x^2 + y^2) = (x - y)(x^2 + xy)$.

(*c*) $x(x + 7) - (x + 3)(x + 4) + 12 = 0$.

(*d*) $x(7 - x) + (3 - x)(4 - x) = 12$.

(*e*) $4x^2 + 7x + 2y = 0$. (*f*) $4x^2 + 7x - 2y = 0$.

(*g*) $x^4 = (x^2 + 1)(x + 1)(x - 1) + 1$.

(*h*) $x^4 = (1 + x^2)(1 + x)(1 - x) + 1$.

(*i*) $(ax - b)^2 + (bx + a)^2 = (a^2 + b^2)(1 + x^2)$.

(*j*) $(ax - b)^2 + (ax + b)^2 = (a^2 + b^2)(1 + x^2)$.

(*k*) $(x - y)^3 + (y - z)^3 + (z - x)^3 = 3(x - y)(y - z)(z - x)$.

(*l*) $(x + y + z)^3 - (x^3 + y^3 + z^3) = 3(x + y)(y + z)(z + x)$.

(*m*) $\dfrac{yz}{(x - y)(x - z)} + \dfrac{zx}{(y - z)(y - x)} + \dfrac{xy}{(z - x)(z - y)} = 1.$

17. Equivalent Equations. Two equations are said to be equivalent when every solution of the first is a solution of the second and conversely, every solution of the second is a solution of the first.

For example, the equations

$$\frac{x}{3} - \frac{5}{7} = 0$$

and

$$7x = 15$$

are equivalent; each has the unique solution $x = 2\frac{1}{7}$.

On the other hand

$$2x - 3 = x - 1$$

and

$$(2x - 3)^2 = (x - 1)^2$$

are not equivalent; the latter has the solution $1\frac{1}{3}$, which does not satisfy the first.

18. Transformations of Equations. The following changes in an equation lead always to an equivalent equation:

1. Transposition of terms with change of sign.

2. Multiplication, or division, of all the terms by the same constant (not zero).

If all the terms of an equation be transposed to the left side (so that the right member is zero), if the left member be factored, and if each of the factors be equated to zero, then the solutions of the separate equations so formed are all solutions of the original equation, and it has no others.

EXAMPLE. The equations

$$\frac{x^3 + 5x}{6} = x^2 - x + 1 \quad \text{and} \quad (x - 1)(x - 2)(x - 3) = 0$$

are equivalent, and the solutions of the latter are seen by inspection to be $x = 1$, $x = 2$, $x = 3$.

The following changes in an equation lead to a new equation which is satisfied by every solution of the given equation, but which generally has other solutions also.

3. Multiplying through by an expression containing unknowns (defined for all values of these unknowns).

4. Squaring both members, or raising both members to the same positive integral power.

Since the new equation is not, in general, equivalent to the given equation, *it is necessary to test all results by substituting them in the given equation in its original form.*

EXAMPLES. Every solution of the equation

$$\frac{x^2}{6} + 1 = \frac{5x}{6}$$

is a solution of the equation

$$x^3 + 6x = 5x^2,$$

which is formed by multiplying the first through by $6x$; but they are not equivalent, since $x = 0$ satisfies the second but does not satisfy the first.

Every solution of the equation

$$\frac{3x - 1}{x + 1} = \frac{x - 1}{x - 2}$$

is a solution of the equation

$$3x^2 - 7x + 2 = x^2 - 1,$$

which results from clearing the former of fractions. These two equations are in fact equivalent. Each is satisfied by $x = \frac{1}{2}$, and by $x = 3$, and by these only.

Every solution of the equation

$$x - 4 = \sqrt{x + 2}$$

is a solution of the equation

$$x^2 - 9x + 14 = 0,$$

which results from squaring and transposition in the former; but they are not equivalent; the latter equation has the two solutions $x = 2$, $x = 7$, while the former has only one, $x = 7$.

19. Simultaneous Equations. When a common solution of two or more equations is sought, the equations are said to be ***simultaneous.*** For example, each of the equations

(8) $$3x - 2y = 4$$

and

(9) $$2x - y = 3$$

has an infinite number of solutions: $(0, -2)$, $(2, 1)$, $(4, 4)$, $(6, 7)$, etc., satisfy (8), and $(1, -1)$, $(2, 1)$, $(3, 3)$, $(4, 5)$, etc., satisfy (9). But $(2, 1)$ is the only common solution.

By a solution of a set of equations is meant a common solution of all the equations of the set, regarded as simultaneous equations. Thus, the set of equations (8) and (9) has a unique solution, namely, $x = 2$, $y = 1$.

Two sets of simultaneous equations are equivalent when each set is satisfied by all of the solutions of the other set.

If each of two or more equations from a set of simultaneous equations be multiplied through by any constant, or by any expression containing unknowns,* and if the resulting equations

*Defined for all values of the unknowns.

be added or multiplied together, the new equation will be satisfied by all the (common) solutions of the given set.

EXAMPLE. If in the set of simultaneous equations,

$$2x^2 + 2y^2 - 3x + y = 9,$$
$$3x^2 + 3y^2 + x - y = 14,$$

we multiply the first by -3, the second by 2, and add, the resulting equation

$$11x - 5y = 1$$

is satisfied by every solution of the given set. One such solution is $x = 1$, $y = 2$.

20. Elimination. By a proper choice of multipliers we can use the above principle to secure a new equation lacking a certain term, or certain terms, which occur in the given set of equations. The missing terms are said to have been eliminated and this process is called ***elimination by addition.***

EXAMPLES. We can eliminate the term in x^2 from the equations,

$$\begin{array}{r||l} -2 & 5x^2 - 9x = 2, \\ 5 & 2x^2 - x = 6, \end{array}$$

by multiplying by -2 and $+5$, respectively, and adding. The result is $13x = 26$. We conclude that if the given equations have a common solution, it is $x = 2$, and we verify that this is a solution of each.

If we eliminate x^2 from the equations,

$$\begin{array}{r||l} -2 & 5x^2 + 9x = 2, \\ 5 & 2x^2 + 5x = 5, \end{array}$$

we obtain

$$7x = 21.$$

Since $x = 3$ is not a solution of the given equations, they have none.

When y is eliminated from the equations,

$$\begin{array}{r||l} 2 & 3x^2 - 4x - 15y + 1 = 0, \\ -3 & 2x^2 - 3x - 10y + 1 = 0, \end{array}$$

the result is

$$x - 1 = 0$$

and on substituting $x = 1$ in either of the given equations, we find $y = 0$. Therefore (1, 0) is the unique common solution.

When it is possible to solve one of a set of simultaneous equations for one of the unknowns, we can eliminate this unknown by substituting the value thus found in the other equations of the set. This is called ***elimination by substitution.***

For example, to eliminate t from the set of equations,

$$x = a(1 + t^2),$$
$$y = a(1 + t),$$

solve the second for t and substitute this value in the first. The result is

$$x = \frac{y^2}{a} - 2y + 2a,$$

which is equivalent to the equation

$$y^2 - ax - 2ay + 2a^2 = 0.$$

If we can solve each of two simultaneous equations for the same unknown, this unknown will be eliminated by equating these two values to each other. This is called ***elimination by comparison.***

Thus, if we solve each of the equations

$$x^2 - xy - 4x + 2y + 1 = 0,$$
$$2x^2 - 2xy + 3x - 2y + 3 = 0,$$

for y, and equate these values, the result is

$$\frac{x^2 - 4x + 1}{x - 2} = \frac{2x^2 + 3x + 3}{2x + 2}$$

which is equivalent to the equation

$$(5x + 8)(x - 1) = 0.$$

21. Linear Equations. An equation of the first degree in the unknown quantities is called a ***linear*** equation. A set of linear simultaneous equations can be solved, if they have a solution, by successively eliminating the unknowns until a single equation in one unknown is obtained.

Examples.

$$\begin{array}{c|l} 3 & 2x + y = 4, \\ 1 & x - 3y = 9. \end{array}$$

Eliminating y by *addition*, we obtain

$$7x + 0 \cdot y = 21.$$

Eliminating x, we get

$$0 \cdot x + 7y = -14.$$

We conclude that if the given equations have a solution it is $x = 3$, $y = -2$, and we verify that this is a solution.

To eliminate x by *substitution* from the equations

$$7x - 9y = 15,$$

$$5x - 8y = 17,$$

solve the first for x and substitute this value in the second. The result is

$$5\left(\frac{9y + 15}{7}\right) - 8y = 17,$$

which is equivalent to $y = -4$. Substituting -4 for y in either of the given equations, we find $x = -3$. Finally, we verify that $x = -3$, $y = -4$, is a solution of the given set.

To eliminate x by *comparison* from the equations

$$3x - 7y = 19,$$

$$2x - 5y = 13,$$

solve each equation for x, and equate the results. This gives

$$\frac{7y + 19}{3} = \frac{5y + 13}{2},$$

which is equivalent to $y = -1$. Substituting this value for y in either of the given equations leads to $x = 4$.

EXERCISES

1. Solve the following equations and determine whether or not the two equations in each pair are equivalent.

(a) $\frac{x+5}{2} - \frac{x+1}{4} = 3, \qquad \frac{x-3}{2x} = \frac{1}{3} - \frac{3x-7}{2x}.$

(b) $\frac{y-7}{5} + 2 = \frac{y+8}{10}, \qquad \frac{2(y-7)}{y^2+3y-28} + \frac{y-2}{y-4} = \frac{y+3}{y+7}.$

(c) $\frac{3t-5}{4} - \frac{9t-7}{12} = \frac{2}{3t}, \qquad \frac{5t+4}{2t} - \frac{11t-2}{6t} = 3.$

(d) $\frac{6x-1}{4} - \frac{8x+3}{10} = \frac{4x-3}{5}, \qquad \frac{3}{4x} + \frac{7}{x} = 15\frac{1}{2}.$

(e) $\frac{5x+1}{2x-3} + 3 = \frac{x}{2x-3}, \qquad \frac{15x^2-5x-8}{3x^2+6x+4} = 5.$

(f) $x - 2 = \sqrt{2x-5}, \qquad \sqrt{6x-x^2} = 3.$

(g) $x - 1 = \sqrt{3x-5}, \qquad x^2 - 5x + 6 = 0.$

(h) $\left(1+\frac{1}{x}\right)\left(1-\frac{1}{x}\right) = \frac{3}{4}, \qquad (x+2)(x-2) = 0.$

(i) $x = 2, \qquad x(x-1) = 2(x-1).$

(j) $2x = 1, \qquad 8x^3 - 12x^2 + 6x = 1.$

2. Solve the following simultaneous equations and determine whether or not the two sets in each pair are equivalent.

(a) $\begin{cases} 3x + 2y = 32, \\ 20x - 3y = 1. \end{cases} \qquad \begin{cases} 7x - y = 1, \\ 9x + 4y = 70. \end{cases}$

Ans. (2, 13). The two sets are equivalent.

(b) $\begin{cases} 3x + 7y = 2, \\ 7x + 8y = -2. \end{cases} \qquad \begin{cases} 2x + 3y = 0, \\ 4x + y = -4. \end{cases}$

(c) $\begin{cases} 6s + 2t = -3, \\ 5s - 3t = -6. \end{cases} \qquad \begin{cases} s + 5t = 3, \\ s + t = 0. \end{cases}$

(d) $\begin{cases} \frac{3}{5}x + y = \frac{1}{4}, \\ \frac{1}{2}x - \frac{5}{7}y = \frac{19}{14}. \end{cases} \qquad \begin{cases} 5x + 4y = 22, \\ 3x + y = 9. \end{cases}$

Ans. These two sets are not equivalent.

(e) $\begin{cases} 3x - 2y = 1, \\ 3x + 4z = 5, \\ 3y + 5z = 4. \end{cases} \qquad \begin{cases} x - y = 1, \\ x + z = 1, \\ y + z = 0. \end{cases}$

Ans. (−1, −2, 2). *Ans.* An infinite number of solutions.

3. Eliminate the x^2 term from the equations

$$x^2 - 2y^2 + 13x + 2y = 1, \qquad 3x^2 + 4y^2 - x + 6y = 3.$$

4. Eliminate y from the equations

$$x^2 + 3xy - x + 1 = 0, \qquad 2x + y + 1 = 0.$$

5. Eliminate t from the equations

$$x = a\frac{3 - t^2}{1 + t^2}, \qquad y = at\frac{3 - t^2}{1 + t^2}.$$

6. Eliminate t from the equations

$$t^2x = t^4 + t^2 + 1, \qquad ty = t^2 - 1.$$

7. Eliminate m from the equations

$$y = \frac{2a}{m} - mx, \qquad x = my.$$

8. Clear the following equations of fractions and radicals and determine in each whether the resulting equation is equivalent to the given one:

(a) $\dfrac{3 - 2x}{2 - 3x} = \dfrac{2x + 1}{3x - 1}$.

(b) $\dfrac{3}{x - 1} - \dfrac{2}{x + 2} = \dfrac{10}{x^2 + x - 2}$.

(c) $\dfrac{x}{4} + \dfrac{4}{x} = \dfrac{x}{9} + \dfrac{9}{x}$.

(d) $\dfrac{x(x + 9)}{x^2 - 9} - \dfrac{3}{x + 3} + 2 = 0$.

(e) $x + \sqrt{x + 6} = 0$.

(f) $\sqrt{6 - 5x} = 2\sqrt{x + 6}$.

(g) $1 + \sqrt{x} = \sqrt{x + 6}$.

(h) $\sqrt{x} + \sqrt{x + 4} = 4$.

(i) $\sqrt{12x + 1} - 2\sqrt{x + 5} = 1$.

(j) $\sqrt{4x - 3} - 2\sqrt{x + 2} + 1 = 0$.

9. How must 1% ammonia and 28% ammonia be mixed to get 12 pints of 10% ammonia?

Ans. 8 and 4 pints.

10. Two given mixtures contain respectively $p\%$ and $q\%$ of a certain ingredient. Show that if x units of the first be combined with y units of the second so that the resulting mixture contains $r\%$ of this ingredient, then $x : y = r - p : q - r$.

10. Assume that gravel has 45% voids and sand 33%, and that 4 bags of cement make 3.8 cu. ft., how much cement, sand, and gravel are necessary to make 1 cu. yd. of concrete?

(*a*) in a 1 : 2 : 4 mixture. (*b*) in a 1 : 3 : 6 mixture.

(*c*) in a 1 : 2 : 3 mixture. (*d*) in a 1 : 3 : 5 mixture.

11. How many pounds of skimmilk must be extracted from 12000 lbs. of 4% milk to raise the test to 4.5%?

Ans. $1333\frac{1}{3}$ lbs.

12. How many pounds each of 40% cream and skimmilk are required to make 125 pounds of 18% cream?

Ans. 56.25 lbs. cream, 68.75 lbs. skimmilk.

13. How many pounds each of 25% cream and 3.5% milk are required to make 130 pounds of 22.5% cream?

Ans. 114.8 lbs. of 25%, 15.2 lbs. of 3.5%.

14. How much 25% cream must be added to 1000 pounds of 50% cream to reduce it to 40% cream?

Ans. $666\frac{2}{3}$ lbs.

15. How many pounds each of 50% and 25% cream must be mixed together to produce 1000 pounds of 40% cream?

Ans. 600 lbs. of 50%, 400 lbs. of 25%.

22. Polynomials. Expressions of the form

$$1 - x,\ x^2 - 3x + 2,\ x + \sqrt{3}x^3 + 3.4 + \tfrac{1}{3}x^2,$$

$$\frac{x^4 - 2x^2 + x - 5}{3},\ 3x^2,$$

are examples of polynomials in x;

$$y - 5 + 4y^2,\ z^5 + \frac{2z}{5} + \sqrt{2}z^2 - \frac{3}{7},\ \frac{a^2}{3} + a^3 - 1 + 2a,$$

are polynomials in y, z, and a, respectively.

A ***polynomial***, in x for example, is a sum of terms each containing a positive integral power of x multiplied by a coefficient independent of x, and usually also an absolute term.

If any number (value of x) be substituted for x, the polynomial reduces to a number called a value of the polynomial.

To each value of x, which is called the variable, there corresponds a unique value of the polynomial. For example, the values of $x^2 - 3x + 2$ which correspond to $x = 0$, $x = 1$, $x = \frac{1}{2}$, are 2, 0, $\frac{3}{4}$.

The ***degree** of any term* in a polynomial is the exponent of the variable in that term. The ***degree** of a polynomial* is the degree of the term of highest degree in it. Polynomials are usually arranged according to the degrees of the terms and it is sometimes convenient to supply with zero coefficients missing terms of degree lower than the degree of the polynomial; thus

$$3x^2 + 0\cdot x + 2, \qquad z^5 + 0\cdot z^4 + 0\cdot z^3 + \sqrt{2}z^2 + \tfrac{2}{5}z - \tfrac{3}{7}.$$

A sharp distinction is to be made between the coefficients and the exponents in a polynomial. The coefficients are very general: they may be any real numbers whatever, natural numbers, rational or irrational numbers, positive, negative, or zero. On the other hand the exponents are very special: they must be positive integers. Thus while the expressions

$$x^2 + 1, \quad y^2 + y/2, \quad z^2 - \pi z + \sqrt{2},$$

are polynomials, the expressions

$$x^{1/2} + 1, \quad y^2 + 2/y, \quad z^2 - 3z + \sqrt{z},$$

are not.

23. Polynomial of the *n*th Degree. A polynomial of degree n in x (n being any given natural number 1, 2, 3, $\cdots$) can be reduced by merely rearranging its terms and adding the coefficients of like powers of x to the form

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n$$

in which the a's (coefficients) are any real numbers ($a_0 \neq 0$ but any or all the others may be zero), and the exponent n is a positive whole number.

24. Linear Equations. An equation of the first degree, or a ***linear*** equation, in one unknown, x for example, is the result of equating to zero a polynomial of the first degree in x,

$$ax + b = 0 \qquad (a \neq 0). \tag{10}$$

This equation has one and only one solution. The method of finding the solution is already known to the student.

25. Quadratic Equations. An equation of the second degree, or a ***quadratic*** equation, in x for example, is the result of equating to zero a polynomial of the second degree in x,

$$ax^2 + bx + c = 0 \qquad (a \neq 0). \tag{11}$$

Any equation which can be reduced to this form by merely transposing and combining like terms is also called a quadratic. Thus,

$$(x - 1)(x - 2) = 6(x - 3)$$

is a quadratic.

Solution by factoring. If the polynomial $ax^2 + bx + c$ can be factored into two linear factors in x (*i. e.*, polynomials of the first degree in x) the roots of the quadratic equation $ax^2 + bx + c = 0$ can be found by inspection.

Example 1. Solve $6x^2 + x = 15$. Transpose all terms to the left side and factor. In order to do this we seek a pair of numbers whose product is 6 and another pair whose product is -15 and such that the cross product is 1. The work may be put down as follows:

$$6x^2 + x - 15 = 0$$

$$\begin{array}{ll} 6 & -5 \\ 1 & +3 \end{array}$$

This gives the cross product 13, but a few trials of other factors and other arrangements quickly leads to the combination

$$\begin{array}{ll} 3 & +5 \\ 2 & -3 \end{array}$$

which gives the cross product 1 as desired. Hence the factors are $3x + 5$ and $2x - 3$, and we have to solve the equation

$$(3x + 5)(2x - 3) = 0.$$

On equating the first factor to zero (mentally) and solving we get $x_1 = -5/3$ and similarly from the second factor $x_2 = +3/2$, and these are the two solutions of the given quadratic equation.

EXAMPLE 2.

$$3x^2 - \tfrac{2}{3}x = \frac{x^2 - 5x}{7}.$$

Transposing and combining terms this reduces to

$$\tfrac{20}{7}x^2 + \tfrac{1}{21}x = 0$$

which factors by inspection into

$$x(\tfrac{20}{7}x + \tfrac{1}{21}) = 0$$

whence $x_1 = 0$ and $x_2 = -\frac{1}{60}$.

If there are fractional coefficients in a quadratic it is usually best to reduce it to an equivalent equation free from fractions by multiplying every term by the least common multiple of all the denominators. Thus in Example 2, we could multiply every term by 21 and obtain,

$$63x^2 - 14x = 3x^2 - 15x.$$

EXERCISES

Solve the following quadratic equations.

1. $2x^2 - 5x = 3$.
2. $10x^2 + x = 2$.
3. $6x^2 + 5x = 6$.
4. $15x^2 - x = 6$.
5. $6x^2 - 5 = 7x$.
6. $28x^2 - 15 = x$.
7. $135x^2 + 3x = 28$.
8. $78x^2 - x = 2$.
9. $3y^2 + y = 10$.
10. $14y^2 + y = 168$.
11. $6y^2 + 11y = 35$.
12. $15y^2 + 4 = 16y$.
13. $6a^2 + a = 5$.
14. $2a + 3 = 8a^2$.
15. $9a(2a + 1) = 14$.
16. $10(2a^2 - 3) + a = 0$.
17. $3(2s^2 - 7) = 5s$.
18. $15(2t^2 - 1) + 7t = 0$.
19. $p(12p - 7) = 10$.
20. $5(3r^2 - 8) + r = 0$.

21. $\frac{8}{17}x^2 + 2x + 1\frac{4}{17} = 0.$ 22. $\frac{y(y-1)}{7} = 3(y+1) - y^2.$

23. $z(z-1) = \frac{2}{11}(6z-1).$ 24. $t^2 + 3t - 1 = \frac{1-t}{6}.$

25. $(1-e^2)x^2 - 2px + p^2 = 0.$

Clear the following equations of fractions, solve the resulting equations and test their solutions in the given equations.

26. $\frac{x+3}{2x-7} = \frac{2x-1}{x-3}.$ 27. $\frac{3}{x^2-1} - \frac{1}{2(x+1)} = \frac{1}{4}.$

28. $\frac{3}{x-5} + \frac{2x}{x-3} = 5.$ 29. $\frac{1}{x-1} - \frac{2}{x-2} = \frac{3}{x-3} - \frac{4}{x-4}.$

26. Solution of a Quadratic by Completing the Suqare. If the polynomial on the left of the quadratic equation

$$ax^2 + bx + c = 0$$

cannot readily be factored by inspection, the equation can be solved by transposing the absolute term c, completing the square of the terms in x and extracting the square roots of both sides.

To complete the square of $ax^2 + bx$ is to find a number d such that $ax^2 + bx + d$ is the square of a linear factor in x and it can always be done as follows: 1) *extract the square root of the first term;* 2) *double this;* 3) *divide this into the second term;* 4) *square the quotient.*

EXAMPLE. Solve $6x^2 - 4x - 1 = 0.$

Transpose -1, and find the number to complete the square of $6x^2 - 4x$ by the above four steps: 1) $x\sqrt{6}$, 2) $2x\sqrt{6}$, 3) $2/\sqrt{6}$, 4) $2/3$; add this to both sides:

$$6x^2 - 4x + \tfrac{2}{3} = \tfrac{5}{3}.$$

Extracting the square roots, we have

$$x\sqrt{6} - \sqrt{\tfrac{2}{3}} = \pm\sqrt{\tfrac{5}{3}},$$

whence solving for x, we find

$$x_1 = \tfrac{1}{3} + \tfrac{1}{6}\sqrt{10}, \qquad x_2 = \tfrac{1}{3} - \tfrac{1}{6}\sqrt{10}.$$

The computations are more easily made, if we multiply the given equation through by a number which will make the coefficient of x^2 a perfect square. In the above example we should have to solve the equivalent equation,

$$36x^2 - 24x + (\quad) = 6.$$

The number required to complete the square is 4,

$$36x^2 - 24x + 4 = 10.$$

Whence

$$6x - 2 = \pm \sqrt{10}$$

and

$$x_1 = \tfrac{1}{3} + \tfrac{1}{6}\sqrt{10}, \qquad x_2 = \tfrac{1}{3} - \tfrac{1}{6}\sqrt{10}.$$

EXERCISES

Solve these equations by completing the square.

1. $4x^2 + 3x = 9$.
2. $25x^2 - 14x + 1 = 0$.
3. $50x^2 + 12x = x^2 - \tfrac{1}{2}$.
4. $(x^2 + 1)\sqrt{3} = 4x$.
5. $12x^2 + 5x = 1$.
6. $6y^2 + 1 = 6y$.
7. $3z^2 = 13(z - 1)$.
8. $x + 2 = 11x(1 - x)$.
9. $12t^2 - 4(a + b)t + ab = 0$.
10. $2(y^2 + c^2) = 5cy$.

27. Solution of a Quadratic by a Formula. By the process of completing the square, a formula for the roots of the general quadratic equation can be found as follows. Given the equation

$$ax^2 + bx + c = 0, \tag{12}$$

multiply through by $4a$, transpose $4ac$, and complete the square,

$$4a^2x^2 + 4abx + b^2 = -4ac + b^2,$$

extracting the square roots, we have

$$2ax + b = \pm \sqrt{b^2 - 4ac},$$

whence we find

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \tag{13}$$

which gives the two roots

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \tag{14}$$

This result may be used as a formula for the solution of any quadratic equation by substituting for a, b, c, of this formula their values from the given equation.

EXAMPLE. Solve $3x^2 + 4x - 15 = 0$.
Here $a = 3$, $b = 4$, $c = -15$, and by the formula

$$x = \frac{-4 \pm \sqrt{16 + 180}}{6},$$

whence

$$x_1 = \frac{-2 + 7}{3} = \frac{5}{3} \quad \text{and} \quad x_2 = \frac{-2 - 7}{3} = -3.$$

EXERCISES

Solve the following equations by the formula.

1. $2x^2 + 3x = 4$.
2. $x^2 = 220 + 9x$.
3. $5x^2 + 3x = 3$.
4. $5x^2 + 5x + 1 = 0$.
5. $15y^2 = 86y + 64$.
6. $5z^2 = 8z + 21$.
7. $a^2 + a = 3$.
8. $p^2 + 3p = 40$.
9. $t^2 + 3a^2 = 4at - 1$.
10. $5m^2 + 21m + 4 = 0$.

Solve the following equations by any method and test all results in the given equation.

11. $(2x - 3)^2 = 8x$.
12. $x^2 - 2\sqrt{3}x + 2 = 0$.
13. $\dfrac{2x}{x+2} + \dfrac{x+2}{2x} = 2$.
14. $\dfrac{x+1}{x} + 1 = \dfrac{x}{x-1}$.
15. $\dfrac{x+1}{x(x-2)} - \dfrac{1}{2x-2} + \dfrac{1}{2x} = 0$.
16. $\dfrac{4}{x-1} - \dfrac{1}{4-x} = \dfrac{3}{x-2} - \dfrac{2}{3-x}$.
17. $3x^2 + (9a - 1)x - 3a = 0$.
18. $x^2 - 2ax + a^2 - b^2 = 0$.
19. $c^2x^2 + c(a - b)x - ab = 0$.
20. $x^2 - 4ax + 4a^2 - b^2 = 0$.
21. $x^2 - 6acx + a^2(9c^2 - 4b^2) = 0$.
22. $(a^2 - b^2)x^2 - 2(a^2 + b^2)x + a^2 - b^2 = 0$.

Solve for y in terms of x.

23. $x^2 + 12xy + 9y^2 + 3 = 0$.
24. $x^2 - 4xy - 4y^2 + x = 0$.
25. $11x^2 + 30xy + 25y^2 = 3$.
26. $8x^2 - 12xy + 4y^2 = x + 1$.
27. $6x^2 - xy - 2y^2 = 0$.
28. $21x^2 = xy + 10y^2$.
29. $30x^2 + 15y^2 = 43xy$.
30. $12x^2 + 41xy + 35y^2 = 0$.
31. $2x^2 + 3xy - 2y^2 + x + 7y - 3 = 0$.
32. $3x^2 + 10xy + 8y^2 + 4x + 2y - 15 = 0$.
33. $10x^2 + 7xy + y^2 - x - 2y - 3 = 0$.
34. $12x^2 = 4xy + 21y^2 + 2x + 29y + 10$.

35. A farmer mows around a meadow 18×80 rods. If the swath averages 5 ft. 6 in., how many circuits will cut half the grass?

Ans. 12.

36. What are eggs worth when 2 more for a quarter lowers the price 5 cents a dozen?

37. If the radius of a circle be divided in extreme and mean ratio the greater part is the side of the regular inscribed decagon. What is the perimeter of the regular decagon inscribed in a circle 2 feet in diameter? *Ans.* 6.180

38. When a heavy body is thrown upward with an initial velocity v ft. per second, its distance from the earth's surface at the end of t seconds is given by the equation $d = vt - 16t^2$. If a projectile is shot upward with a muzzle velocity of 1000 ft. per second, when will it be 15,600 ft. high? *Ans.* 30 and $32\frac{1}{2}$ sec.

28. Equations in Quadratic Form. The terms of an equation which is not a quadratic in the unknown can sometimes be grouped so as to make it a quadratic in an expression containing the unknown. Thus, $x^4 - 13x^2 + 36 = 0$ is not a quadratic in x but it is a quadratic in x^2; again if the terms of $x^4 - 6x^3 + 7x^2 + 6x = 8$ be grouped in the form

$$(x^4 - 6x^3 + 9x^2) - 2(x^2 - 3x) = 8$$

it is seen to be a quadratic in $(x^2 - 3x)$.

EXAMPLE 1. Solve $6x - 7\sqrt{x} = 20$.

Transpose 20 and this can be solved by the formula as a quadratic in $\sqrt{x}$; whence,

$$\sqrt{x} = \frac{7 \pm \sqrt{49 + 480}}{12}$$

and, since the positive square root cannot be negative,

$$\sqrt{x} = 2.5 \qquad \text{and} \qquad x = 6.25.$$

We verify that this satisfies the given equation.

EXERCISES

1. $x^4 - 13x^2 + 36 = 0$.
2. $x + \sqrt{x + 6} = 14$. *Ans.* 10.
3. $2x^2 + 3\sqrt{x^2 - 2x + 6} = 4x + 15$. *Ans.* -1 and 3.
4. $x + \sqrt{1 - x^2} + \dfrac{1}{x + \sqrt{1 - x^2}} = 2$. *Ans.* 0 and 1.
5. $\sqrt[3]{x} + 18 = 5\sqrt[3]{x^2}$. *Ans.* 8 and $-729/125$.
6. $x^4 - 6x^3 + 7x^2 + 6x = 8$. *Ans.* $-1, 1, 2, 4$.

29. Imaginary Roots. . There are quadratics which are not satisfied by any real number. For example, $x^2 = -4$, $x^2 + 2x + 2 = 0$. This is because the square of every real number (except 0) is positive. If we attempt to solve the equation $x^2 + 2x + 2 = 0$ either by completing the square or by the formula we are led to the indicated square root of a negative number, and this is not a real number; thus

$$x = \frac{-2 \pm \sqrt{4 - 8}.}{2}$$

These, and other considerations have led to the invention of numbers whose squares are negative real numbers; they are called *imaginary* numbers. The imaginary unit is usually denoted by i. By definition, we have

$$i^2 = -1, \qquad i^3 = i \cdot i^2 = -i, \qquad i^4 = i \cdot i^3 = +1,$$
$$i^5 = i \cdot i^4 = i, \qquad i^6 = -1, \qquad i^7 = -i, \qquad i^8 = +1, \qquad \text{etc.}$$

The number $r \cdot i$, where r is any real number is called a pure imaginary number; e. g., $2i$, $5i$, $-3i$, $-\frac{1}{2}i$, $i\sqrt{3}$, etc. The squares of pure imaginary numbers are negative real numbers; e. g., $(2i)^2 = 2^2 i^2 = -4$; $(-3i)^2 = (-3)^2 i^2 = -9$; $(i\sqrt{3})^2 = -3$.

Conversely, the square roots of negative real numbers are imaginary numbers; the square roots of -4 are $2i$ and $-2i$; i. e., $\sqrt{-4} = i\sqrt{4} = 2i$, $-\sqrt{-4} = -i\sqrt{4} = -2i$; $\sqrt{-3} = i\sqrt{3}$, $-\sqrt{-3} = -i\sqrt{3}$; in general, $\sqrt{-p} = i\sqrt{p}$, where p is a real positive number.

Expressions of the form $2 + 5i$, $1 - i$, $3 - 2i$, $-1 + i$, etc., indicating the sum of a real and an imaginary number are called ***complex numbers.*** They may be added, subtracted, multiplied, and divided by the laws of algebra as though i were a real number and the results simplified by putting -1 for i^2, $-i$ for i^3, $+1$ for i^4, etc.

We can now say that every quadratic equation can be solved. The solutions of the equation

$$x^2 - 2x + 2 = 0$$

may be found by the formula,

$$x = \frac{2 \pm \sqrt{4 - 8}}{2}$$

whence

$$x_1 = 1 + i \qquad \text{and} \qquad x_2 = 1 - i;$$

and we verify both these answers as follows:

$$(1 + i)^2 - 2(1 + i) + 2 = 0, \qquad (1 - i)^2 - 2(1 - i) + 2 = 0.$$

EXERCISES

1. $x^2 - 4x + 5 = 0$.
   Ans. $2 \pm i$.
2. $x^2 + 6x + 13 = 0$.
   Ans. $-3 \pm 2i$.
3. $36x^2 - 36x + 13 = 0$.
   Ans. $1/2 \pm i/3$.
4. $2x^2 + 2x + 1 = 0$.
   Ans. $-1/2 \pm i/2$.

5. $x^2 + 4 = 0$.
Ans. $\pm 2i$.
6. $x^2 + x + 1 = 0$.
Ans. $-1/2 \pm i\sqrt{3}/2$.
7. $x^2 - 2x + 3 = 0$.
8. $x^2 - \frac{2}{3}x + 1 = 0$.
9. $x^2 - 2x\sqrt{3} + 7 = 0$.
10. $2x^2 - 2x + 5 = 0$.
11. $x^2 + 3x + 2.5 = 0$.
12. $49x^2 - 56x + 19 = 0$.

30. The Sum and Product of the Roots. The two roots of the quadratic equation

$$ax^2 + bx + c = 0$$

are by (14), § 27,

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

The ***sum*** of these roots is $-b/a$, and their ***product*** is $+c/a$, as may be seen by adding and multiplying them together.

We can thus find the sum and the product of the roots of a given quadratic equation without solving it. Thus in the equation,

$$36x^2 - 36x + 13 = 0$$

the sum of the roots is 1, and their product is 13/36.

Again in the equation,

$$my^2 - 4ay + 4ab = 0$$

the sum of the roots is $4a/m$, and their product is $4ab/m$.

31. Equation having Given Roots. We have seen that if the left member of the quadratic equation

$$ax^2 + bx + c = 0$$

can be separated into linear factors, its roots can be found by inspection. Therefore if we wish to make up a quadratic equation whose roots shall be two given numbers, r and s for example, we have only to write

$$a(x - r)(x - s) = 0$$

and multiply out. The factor a is arbitrary and may be chosen so as to clear the equation of fractions if desired; thus, to make an equation whose roots shall be $\frac{3}{2}$ and $-\frac{2}{5}$, we write,

$$a(x - \tfrac{3}{2})(x + \tfrac{2}{5}) = 0,$$

and if we take $a = 10$, the resulting equation is

$$10x^2 - 11x - 6 = 0.$$

32. Number of Roots. Conversely, it is readily shown that if r and s are roots of the quadratic equation,

$$ax^2 + bx + c = 0$$

then the left member can be factored in the form,

$$a(x - r)(x - s) = 0 \tag{15}$$

and this shows that no quadratic can have more than two roots. Some quadratics have only one root; for example $4x^2 + 9 = 12x$ is satisfied only by $x = 3/2$.

If $b^2 - 4ac = 0$, then the polynomial $ax^2 + bx + c$ is a perfect square and the equation $ax^2 + bx + c = 0$ has only one root, and conversely.

For, if $b^2 - 4ac = 0$, then $c = b^2/4a$ and this is precisely the number necessary to complete the square of $ax^2 + bx$. Also if $b^2 - 4ac = 0$, the formula (13), § 27, gives not two but one root.

33. Kind of Roots. If a, b, and c, are real numbers and if $b^2 - 4ac > 0$, then the quadratic equation $ax^2 + bx + c = 0$ has two real roots; but if $b^2 - 4ac < 0$, the equation has two imaginary roots.

This is seen at once on noting the formula (13), § 27, which gives the roots.

EXAMPLE 1. $4x^2 - 12x + 9 = 0$.

Here $b^2 - 4ac = 144 - 144 = 0$, the left member is a perfect square and the equation has only one root.

EXAMPLE 2. $3x^2 - 5x + 2 = 0$.

Compute $b^2 - 4ac = 25 - 24 = +1$, which shows that the equation has two real roots.

EXAMPLE 3. $x^2 + x + 1 = 0$.

Here $b^2 - 4ac = -3$, which shows that the equation has imaginary roots.

If a, b, and c, are rational numbers, then the roots of the equation $ax^2 + bx + c = 0$ are rational if $b^2 - 4ac$ is a perfect square, i. e., the square of a rational number; in particular if a, b, and c, are integers and if $b^2 - 4ac$ is a perfect square the left member of the equation can be factored by inspection.

EXAMPLE 4. $2x^2 - x - 6 = 0$.

Here $b^2 - 4ac = 1 + 48 = 49$; the left member factors into

$$(2x + 3)(x - 2) = 0$$

whence the roots are $-3/2$ and 2.

EXERCISES

1. Form the equations whose roots are

(a) 1, 3, -5.
(b) $-2, 3, -4, 6$.
(c) $1/3, -7/2, 3/5$.
(d) $\pm 1, \pm 4$.
(e) $\pm\sqrt{2}, \pm\sqrt{5}$.
(f) $0, -2, \pm\sqrt{-2}$.
(g) $3, 5 \pm \sqrt{5}$.
(h) $4 \pm \sqrt{3}, -1 \pm \sqrt{6}$.
(i) $-\alpha, -\beta, -\gamma$.
(j) $k\alpha, k\beta, k\gamma$.
(k) $\alpha + k, \beta + k\ \gamma + k$.
(l) $1/\alpha, 1/\beta, 1/\gamma$.
(m) α, β, γ.
(n) $\alpha^2, \beta^2, \gamma^2$.
(o) $\alpha - \beta, \beta - \gamma, \gamma - \alpha$.
(p) $\alpha\beta, \beta\gamma, \gamma\alpha$.

2. Determine the nature of the roots of the following equations.

(a) $3x^2 - 4x - 1 = 0$.
(b) $5x^2 + 6x + 1 = 0$.
(c) $2x^2 + x - 6 = 0$.
(d) $x^2 - 2x - 1 = 0$.
(e) $5x^2 - 6x + 5 = 0$.
(f) $x^2 - 6\sqrt{3}x - 5 = 0$.
(g) $x^2 + x + 1 = 0$.
(h) $13x^2 - 6\sqrt{3}x + 7 = 0$.
(i) $3x^2 + 2x + 1 = 0$.
(j) $2x^2 - 16x + 9 = 0$.
(k) $5x^2 - 12x - 8 = 0$.
(l) $6x^2 + 4x - 5 = 0$.

(*m*) $5x + 7 = (3x + 2)(x - 1)$. (*n*) $5(x^2 + x + 1) = 1 - 16x$.
(*o*) $2x(x - 3) = 7(3x + 2)$. (*p*) $7(x^2 + 5x + 3) = x(1 - x)$.
(*q*) $3x(x + 1) = (3 - x)(3 + x)$. (*r*) $3x^2 = 13(x - 1)$.
(*s*) $(y + 2)(y - 2) = 2y - 7$. (*t*) $3(y + 1)(y - 1) = 4y$.
(*u*) $6y(3 - 5y) = 19(y - 1)^2$. (*v*) $y^2 - 2y\sqrt{3} + 7 = 0$.

3. Without solving find the sum and the product of the roots of each of the equations in Ex. 2.

4. Determine the nature of the roots of the following equations in which a, b, c are known real numbers.

(*a*) $(x - a)^2 = b^2 + c^2$. (*b*) $(x + a)^2 = 8b^2$.
(*c*) $a(ax^2 + 2bx - a) = b(bx^2 - 2ax - b)$.
(*d*) $y^2 = 2a(y - b) + 2b(y - a)$.
(*e*) $(a + b - c)y^2 - 2cy = (a + b + c)$.
(*f*) $(a + b - c)x^2 + 4(a + b)x + (a + b + c) = 0$.
(*g*) $(b + c - 2a)x^2 + (c + a - 2b)x + (a + b - 2c) = 0$.

5. Determine values of a for which each of the following quadratic equations will have equal roots.

(*a*) $x(x + 4) + 2a(2x - 1) = 0$. (*b*) $(x - 1)^2 = 2a(3x - 7) - 20$.
(*c*) $(x - a)^2 = a^2 - 8a + 15$. (*d*) $x^2 - 15 = 2a(x - 4)$.
(*e*) $3(x^2 + 3ax - a) = x$. (*f*) $9x^2 + 6(a - 4)x + a^2 = 0$.
(*g*) $3ax(x - 1) = ax - 2$. (*h*) $(a+1)x^2-(a + 2)x + \frac{2}{3}a = 0$.
(*i*) $(4a^2 + 3)x^2 + 8a(3 - 2a)x + 4(4a^2 - 12a - 3) = 0$.

6. Find a value of k such that the sum of the roots of the equation $x^2 - 3(k + 1)x + 9k = 0$ shall be one half their product.

7. Construct equations whose roots shall be greater by 2 (also less by 2) than the roots of the equations in Ex. 2.

8. Construct equations whose roots shall be twice (also half) the roots of the equations in Ex. 2.

CHAPTER III

GRAPHIC REPRESENTATION

34. Graphic Methods. The methods studied in plane geometry for constructing various figures when certain of their dimensions and angles are known are used extensively in making designs for machines, plans for buildings and various other structures, and also for solving problems that require the determination of unknown dimensions, angles, areas, etc.

These methods often give the desired results with sufficient accuracy for practical purposes, and they are more direct and rapid than numerical computation. Of even greater importance however is their use in checking the results of calculations, since there are always possibilities of error even when great care is exercised. It should be emphasized that every practical calculation (*i.e.* one which is to be used in construction, or other actual work where time, material, and money will be wasted if the calculation is incorrect) should always be checked by some independent means.

Two rectilinear figures are ***similar*** if their corresponding angles are equal and their corresponding dimensions are proportional.

35. Drawing to Scale. When two plane figures are similar, each is said to be a ***scale drawing*** of the other. The smaller is said to be reduced or drawn to a smaller scale. For example, if a drawing be made of a floor plan of a house so that the angles in the drawing are equal to those in the house itself, and the dimensions of the drawing are $\frac{1}{48}$ of those of the house, it is said to be drawn to a scale of $\frac{1}{4}$ inch to one foot. From such a drawing the builder can read off on a scale divided into quarter inches the dimensions of the parts he is about to construct.

This method of drawing figures to scale, reading off their unknown angles on a protractor, and their unknown dimensions on a conveniently divided scale, furnishes a graphic solution of many problems and it has many practical applications.

Example. The distance $AB = 98$ yards, Fig. 1, and the angles $PAB = 51°$, $PBA = 63°$, having been measured from one side of a river, the triangle can be drawn to scale and the width PR of the river can be read off on the scale, about 75 yards.

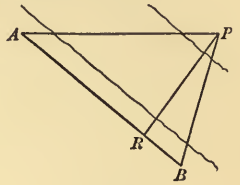


Fig. 1

EXERCISES

1. Find the length of the projection of the altitude of an equilateral triangle upon one of its sides. *Ans.* .75*s*

2. Draw two diagonals through the centre of a regular hexagon. Find the length of the projection of one of them upon the other. *Ans.* *s*.

3. Draw two diagonals through the same vertex of a regular pentagon. Find the length of a projection of one of them upon the other. *Ans.* 1.3*s*

4. The *pitch of a roof* is the ratio of the height of the ridge above the plates to the distance between the plates. Find the length of the rafters and their inclination for a $\frac{1}{2}$ pitch roof on a building 28 ft. wide. *Ans.* 21.8, 50°.

5. Find the length of the corner rafters, and also of the middle rafter on each side of a square roof on a house 34 × 34 feet, the apex of the roof being 12 feet above the top floor. Find also their inclinations. *Ans.* 26.9, 20.8, 26°.5, 35°.

6. The roof of a building 36 ft. wide is inclined at an angle of 54° to the horizon. Find the length of the rafters, allowing 2 ft. overhang. *Ans.* 32.6 ft.

7. To determine the horizontal distance between two points P and Q on the same level but separated by a hill, a point R is selected from which P and Q are visible. Then $PR = 200$ ft., $QR = 223$ ft., and angle $PRQ = 62°$ are measured. Draw the figure and scale off PQ. *Ans.* 210.

8. The steps of a stairway have a tread of 10 in., and a rise of 7 in. Find the inclination of the stringers to the floor. *Ans.* 35°.

9. Plot four points on a sheet of paper. Mark them A, B, C, D. Construct a point P one-half the way from A to B, a point Q one-third the way from P to C, and a point R one-fourth the way from Q to D. Mark the four given points in some other order and repeat the construction. What conclusion do you draw?

36. Rectangular Coördinates of a Point in a Plane. The position of any point in the plane is uniquely determined as

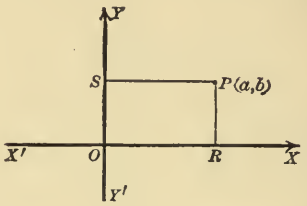


FIG. 2

soon as we know its *distance* and *sense* from each of the two perpendicular lines $X'X$ and $Y'Y$. These lines are taken first, and are drawn in any convenient position.

The distance from $X'X$ ($RP = b$ in the figure) is called the ***ordinate*** of the point P. The distance from $Y'Y$ ($SP = a$ in the figure) is the ***abscissa*** of P.

Abscissas measured to the *right* of $Y'Y$ are *positive*, those to the *left* of $Y'Y$ are *negative*. Ordinates measured *above* $X'X$ are *positive*, those *below negative*.

The abscissa and ordinate taken together are called the ***coördinates*** of the point, and are denoted by the symbol (a, b). In this symbol it is agreed that the number written first shall stand for the abscissa.

The lines $X'X$ and $Y'Y$ are called the ***axes of coördinates***, $X'X$ being the ***axis of abscissas*** or the ***axis of X***, and $Y'Y$ the

axis of ordinates or the ***axis of Y;*** and the point O is called the ***origin*** of coördinates.

The axes of coördinates divide the plane into four parts called quadrants. Figure 3 indicates the proper signs of the coördinates in the different quadrants.

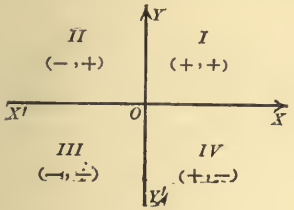


Fig. 3

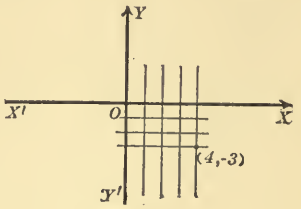


Fig. 4

37. Plotting Points. To plot a point is to locate it with reference to a set of coördinate axes. The most convenient way to do this is to first count off from O along $X'X$ a number of divisions equal to the abscissa, to the right or left according as the abscissa is positive or negative. Then from the point so determined count off a number of divisions equal to the ordinate, upward or downward according as the ordinate is positive or negative. The work of plotting is much simplified by the use of *coördinate paper*, or *squared paper*, which is made by ruling off the plane into equal squares, the sides being parallel to the axes. Thus, to plot the point $(4, -3)$, count off four divisions from O on the axis of X to the right, and then three divisions downward from the point so determined on a line parallel to the axis of Y, as in Fig. 4.

If we let both x and y take on every possible pair of real values, we have a point of the plane corresponding to each pair of values of (x, y). Conversely, to every point of the plane corresponds a pair of values of (x, y).

EXERCISES

1. Plot the following points (3, 3), (4, 5), (− 2, 3), (− 4, − 2), (7, − 2), (0, 4), (0, − 4), (3, 0), (− 3, 0), (0, 0).

2. What is the y-coördinate of any point on the x-axis?

3. What is the x-coördinate of any point on the y-axis?

4. Show that the line joining (5, 4) and (− 5, − 4) is bisected by the origin.

5. Find the distance from the origin to each of the points in Ex. 1.

6. Find the lengths of the sides of the triangle whose vertices are (1, 1), (5, 2), (3, 4). *Ans.* $\sqrt{17}$; $\sqrt{13}$; $2\sqrt{2}$.

7. What is the abscissa of any point upon a straight line parallel to the y-axis and four units to its right?

8. What is the ordinate of any point upon a straight line parallel to the x-axis and three units above it?

9. (*a*) What relation exists between the coördinates of any point of a line bisecting the angle between the positive directions of the two axes? (*b*) Between the positive direction of the y-axis and the negative direction of the x-axis?

10. What relation would exist between the coördinates of any point of the line in Ex. 9 (*a*), if it were raised four units parallel to itself? If it were lowered five units?

38. Statistical Data. The following table shows the rainfall in inches, as observed at the Agricultural Experiment Station at LaFayette, Indiana, by months for 1916, 1917, and the average for the past 30 years.

	Jan.	Feb.	Mar.	Apr.	May	Jun.	Jul.	Aug.	Sep.	Oct.	Nov.	Dec.
1916 . .	7.40	1.16	1.08	1.57	5.82	5.27	3.56	1.81	2.22	2.25	2.25	4.79
1917 . .	1.54	1.25	4.09	4.32	4.75	5.41	1.47	4.09	1.03	5.22	0.13	0.68
Average .	3.11	2.88	3.78	3.38	4.05	3.75	3.54	3.32	3.03	2.46	3.23	2.71

While it is possible by a study of this table to compare the rainfall month by month in the same year, or for the same month in the two years, or any month with the normal for that month,

these comparisons are more easily made and the facts are presented much more emphatically by the diagram shown in Fig. 5. This is made from the data of the table as follows. The 24 vertical lines represent the months of the two-year period. The altitudes of the horizontal lines represent inches of rainfall. The height (ordinate) of the point marked on any vertical line shows the rainfall for that month. The points are connected by lines to aid the eye in following the march of the rainfall. The full line represents the rainfall for 1916 and 1917, the dotted line the normal rainfall as shown by the experience of 30 years.

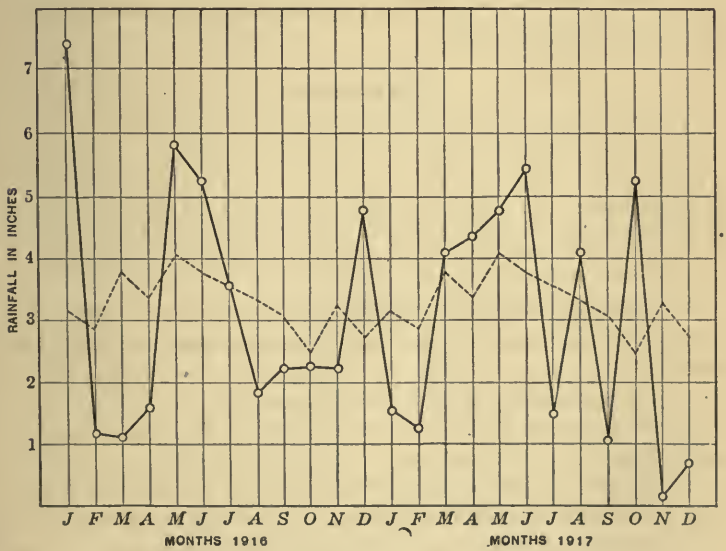


FIG. 5

Rainfall is a discontinuous phenomenon. Moisture is not precipitated continuously, but intermittently. However, if we make a similar diagram showing the temperature at each hour of the day we might have inserted many other points. We

think of the change in temperature as a continuous phenomenon; *e.g.* when the temperature rises from 42° at 8 A.M. to 51° at 9 A.M., we think of it as having passed through every intervening degree in that hour. Thus we can think of the points which represent the temperature on the diagram from instant to instant as lying thick on a continuous curved line. This curve is called the temperature curve.

In making a graph of a discontinuous function like rainfall, we connect the points with straight lines as in Fig. 5, but in case of a continuous function like temperature, a *smooth* curve which passes through all the plotted points is the best graphic representation of the function.

EXERCISES

1. Make a temperature graph from the following data,

Hour, A. M.	12	1	2	3	4	5	6	7	8	9	10	11	12
Temperature	45	45	45	45	43	42	41	40	42	51	57	59	62
Hour, P. M.	1	2	3	4	5	6	7	8	9	10	11	12	..
Temperature	66	70	74	76	76	75	74	73	72	70	69	68	..

2. Determine from Fig. 5 which were the dry months in 1916. In 1917. To what extent do they agree with each other and with the normal?

3. Do as directed in Ex. 2 for the wet months.

4. What straight line in Fig. 5 would represent the average monthly rainfall for 1916? For 1917? For the past 30 years?

5. To what extent does the dotted line in Fig. 5 enable you to predict the probable rainfall in any given month subsequent to 1917?

6. Procure the census data and plot the population graph of the United States by decades for a century.

7. Plot a graph of the attendance of students at your college or University for as many years back as you can secure the data.

8. The following data give the Chicago price per bu. of No. 2 corn by months from Jan., 1903, to May, 1908. Plot the data using years as abscissas and price as ordinates.

	Jan.	Feb.	Mch.	Apr.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.
1903	43	42	41	41	44	47	49	50	45	43	41	41
1904	42	46	49	46	47	53	47	51	51	50	50	43
1905	41	42	45	46	48	51	53	53	51	50	45	42
1906	42	41	39	43	47	50	49	48	47	44	44	40
1907	39	43	43	44	49	51	52	54	60	55	55	57
1908	59	56	58	65	70	....	....	....	....	....	....	....

9. Find from the graph that month in each year in which the highest price occurred. The lowest price. Find the difference for each year between the highest and lowest price for that year. Does there appear to be any relation between these prices and the period of harvest?

10. The following data gives the Chicago price of No. 2 oats by months from Jan., 1903 to May, 1908. Plot the data using years as abscissas and price as ordinates.

	Jan.	Feb.	Mch.	Apr.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.
1903	31	33	31	32	33	35	33	33	35	34	33	34
1904	36	39	38	36	39	39	38	31	29	28	29	28
1905	29	29	29	28	28	30	27	25	25	27	29	29
1906	31	29	28	30	32	33	30	29	30	32	33	33
1907	$33\frac{3}{4}$	37	39	41	44	41	41	44	51	45	44	46
1908	48	48	52	51	53	....	....	....	....	....	....	....

11. Handle the data in Ex. 10 as directed in Ex. 9.

12. A restaurant keeper finds that if he has G guests a day his total daily expenditure is E dollars and his total daily receipts are R dollars. The following numbers are averages obtained from the books:

G.....	210	270	320	360
E.....	16.70	19.40	21.60	23.40
R.....	15.80	21.20	26.40	29.80

Plot two curves on the same set of axes in each case using G as abscissas. For one curve use E as ordinates, for the other use R as ordinates.

Below what value of G does the business cease to be profitable? Connect the points (G, E) by a smooth curve. Continue this curve until it cuts the line $G = 0$. What is the meaning of the ordinate E for $G = 0$? Through what point ought the curve connecting the points (G, R) to pass? *Ans.* (0, 0).

13. The population of the United States by decades was as follows. Plot, and estimate the population for 1920.

Year.	Population.	Year.	Population.	Year.	Population.
1790....	3,929,214	1840....	17,069,453	1890....	62,669,756
1800....	5,308,433	1850....	23,191,876	1900....	76,295,200
1810....	7,229,881	1860....	31,443,321	1910....	91,972,266
1820....	9,663,822	1870....	38,558,371	.....	.....
1830....	12,806,020	1880....	50,155,783	.....	.....

14. The football accidents for the years given are as follows:

Year.	Deaths.	Injuries.	Year.	Deaths.	Injuries.
1901....	7	74	1907....	15	166
1902....	15	106	1908....	11	304
1903....	14	63	1909....	30	216
1904....	14	276	1910....	22	499
1905....	24	200	1911....	11	178
1906....	14	160	.....	.....	.....

Plot two curves, using the years as abscissas and the deaths and injuries respectively as ordinates.

15. The monthly wages in dollars of a man for each of his first 13 years of work was as follows: 28, 30, 37.50, 45, 60, 65, 90, 95, 95, 137, 162, 190, 210. Plot the curve showing the change. Estimate his salary for the fourteenth and fifteenth years. Can you be certain of his salaries for these years?

16. Of 100,000 persons born alive at the same time the following table shows the number dying in the respective age intervals:

Months.	Deaths.	Months.	Deaths.
0–1	4,377	6– 7	579
1–2	1,131	7– 8	533
2–3	943	8– 9	492
3–4	801	9–10	456
4–5	705	10–11	421
5–6	635	11–12	389

Years.	Deaths.	Years.	Deaths.
0– 1	11,462	19– 20	344
1– 2	2,446	29– 30	479
2– 3	1,062	39– 40	644
3– 4	666	49– 50	873
4– 5	477	59– 60	1,404
5– 6	390	69– 70	1,974
6– 7	327	79– 80	1,854
7– 8	274	89– 90	571
8– 9	234	99–100	25
9–10	204	106–107	1

Plot the above data. Make two graphs. In each graph use deaths as ordinates; in one use months as abscissas, in the other use years. When is the ordinate smallest? largest? Does a small ordinate for the years 99–100 and 106–107 indicate a low death rate? Explain. Note the continuous decrease in the ordinate of the first curve.

17. Using the data below and on p. 46, plot a curve using years as abscissas and price of corn as ordinates. Do you notice any regularity in the number of years elapsing between successive high prices? successive low prices? Draw like graphs for the other crops listed?

18. Plot the prices for the yrs. 74, 81, 87, 90, 94, 01, 08, 11, 1916. What do you observe from this curve as to the tendency in the high price of corn? Do you observe any tendency in the lowest prices of corn that is in the prices for the yrs. 72, 78, 84, 89, 96, 02, 06, 1910?

AVERAGE FARM PRICE DECEMBER FIRST

Data from the year book of the Department of Agriculture 1916

Year.	Corn.	Wheat.	Oats.	Barley.	Rye.	Potatoes.	Hay, $ per Ton.
1870.	49.4	94.4	39.0	79.1	73.2	65.0	12.47
1871.	43.4	114.5	36.2	75.8	71.1	53.9	14.30
1872.	35.3	111.4	29.9	68.6	67.6	53.5	12.94
1873.	44.2	106.9	34.6	86.7	70.3	65.2	12.53
1874.	58.4	86.3	47.1	86.0	77.4	61.5	11.94
1875.	36.7	89.5	32.0	74.1	67.1	34.4	10.78
1876.	34.0	97.0	32.4	63.0	61.4	61.9	8.97
1877.	34.8	105.7	28.4	62.5	57.6	43.7	8.37
1878.	31.7	77.6	24.6	57.9	52.5	58.7	7.20
1879.	37.5	110.8	33.1	58.9	65.6	43.6	9.32

Continued on p. 46.

AVERAGE FARM PRICE, DECEMBER FIRST

Continued.

Year.	Corn.	Wheat.	Oats.	Barley.	Rye.	Potatoes.	Hay, $ per Ton.
1880.	39.6	95.1	36.0	66.6	75.6	48.3	11.65
1881.	63.6	119.2	46.4	82.3	93.3	91.0	11.82
1882.	48.5	88.4	37.5	62.9	61.5	55.7	9.73
1883.	42.4	91.1	32.7	58.7	58.1	42.2	8.19
1884.	35.7	64.5	27.7	48.7	51.9	39.6	8.17
1885.	32.8	77.1	28.5	56.3	57.9	44.7	8.71
1886.	36.6	68.7	29.8	53.6	53.8	46.7	8.46
1887.	44.4	68.1	30.4	51 9	54.5	68.2	9.97
1888.	34.1	92.6	27.8	59.0	58.8	40.2	8.76
1889.	28.3	69.8	22.9	41.6	42.3	35.4	7.04
1890.	50.6	83.8	42.4	62.7	62.9	75.8	7.87
1891.	40.6	83.9	31.5	52.4	77.4	35.8	8.12
1892.	39 4	62.4	31.7	47.5	54.2	66.1	8.20
1893.	36.5	53.8	29.4	41.1	51.3	59.4	8.68
1894.	45.7	49.1	32.4	44.2	50.1	53.6	8.54
1895.	25.3	50.9	19.9	33.7	44.0	26.6	8.35
1896.	21.5	72.6	18.7	32.3	40.9	28.6	6.55
1897.	26.3	80.8	21.2	37.7	44.7	54.7	6.62
1898.	28.7	58.2	25.5	41.3	46.3	41.4	6.00
1899.	30.3	58.4	24.9	40.3	51.0	39.0	7.27
1900.	35.7	61.9	25.8	40.9	51.2	43.1	8.89
1901.	60.5	62.4	39.9	45.2	55.7	76.7	10.01
1902.	40.3	63.0	30.7	45.9	50.8	47.1	9.06
1903.	42.5	69.5	34.1	45.6	54.5	61.4	9.07
1904.	44.1	92.4	31.3	42.0	68.8	45.3	8.72
1905.	41.2	74.8	29.1	40.5	61.1	61.7	8.52
1906.	39.9	66.7	31.7	41.5	58.9	51.1	10.37
1907.	51.6	87.4	44.3	66.6	73.1	61.8	11.68
1908.	60.6	92.8	47.2	55.4	73.6	70.6	8.98
1909.	57.9	98.6	40.2	54.0	71.8	54.1	10.50
1910.	48.0	88.3	34.4	57.8	71.5	55.7	12.14
1911.	61.8	87.4	45.0	86.9	83.2	79.9	14.29
1912.	48.7	76.0	31.9	50.5	66.3	50.5	11.79
1913.	69.1	79.9	39.2	53.7	63.4	68.7	12.43
1914.	64.4	98.6	43.8	54.3	86.5	48.9	11.12
1915.	57.5	92	36.1	51.7	83 9	61.6	10.70
1916.	88.9	160.3	52.4	88.2	122.1	146.1	10.59

39. Other Graphic Methods. The statistical data given in the preceding articles has been studied by means of curves or graphs drawn on rectangular cross-section paper. There are other important methods of representing statistical data. Of these methods we will give names to three:

(1) *Bar diagrams* or columnar charts.

(2) *Dot diagrams.*

(3) *Circular diagrams.*

These methods are best explained by means of examples.

40. Bar Diagrams. Below is given a bar diagram or chart comparing the average size of farms for the years 1900 and 1910 for the states indicated.

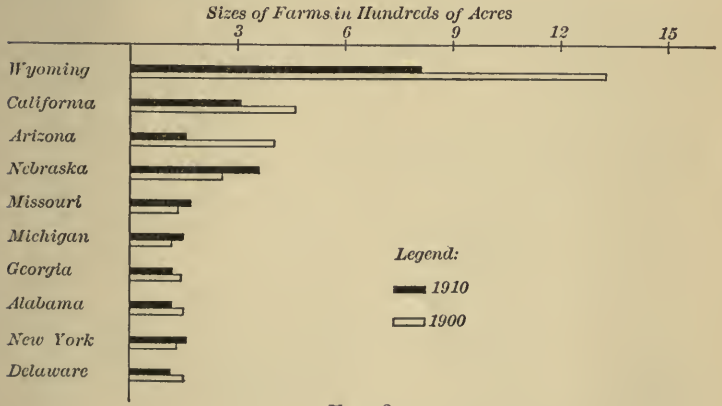


FIG. 6

EXERCISES

Make a bar diagram from the following data:

1. The number of cattle in millions on farms for 1900 and 1910 in the following states were as follows.

Date.	Texas.	Iowa.	Nebraska.	New York.	Oklahoma.	Indiana.
1910...	7.0	4.5	2.9	2.4	2.0	1.3
1900...	10.0	5.5	3.2	2.6	3.3	1.7

2. The sheep on farms in millions in 1910 and 1900 were as follows.

Date.	Texas.	Iowa.	Nebraska.	New York.	Oklahoma.	Indiana.
1910...	1.6	1.1	0.3	0.9	0.1	1.3
1900...	1.8	1.0	0.5	1.7	0.15	1.7

3. Make a bar diagram comparing the number of hours work required by hand and machine labor in producing selected units (U. S. labor bulletin 54).

Description of Unit.	Number of Hours Worked.	
	Hand.	Machine.
Corn 50 bu. husked. Stalk left.....	48.44	18.91
Seed 1000 lbs. cotton.....	223.78	78.70
Harvesting and baling 8 tons timothy....	284.00	92.63
Wheat 50 bu.....	160.63	7.43
Potatoes 500 bu.....	247.54	86.36
Butter 500 lbs. in tubs.....	125.00	12.50
5000 cotton flour sacks.....	137.50	28.33
Quarry 100 tons limestone.....	115.28	80.67
Mine 50 tons bituminous coal.....	171.05	94.30

4. Make a bar diagram comparing the value of farm property for the two years 1900 and 1910.

Year	1910.	1900.
Land.....	28,475,674,169	13,058,007,995
Buildings.....	6,325,451,528	3,556,639,496
Implements and machinery.....	1,265,149,783	749,775,970
Domestic animals, poultry, and bees	4,925,173,610	3,075,477,703

5. Make a bar diagram of the population of the following states for the years 1900 and 1910.

Date.	Colorado.	Nevada.	Idaho.	Washington.	Oregon.	California.
1910.....	799,024	81,875	325,594	1,141,990	672,765	2,377,549
1900.....	539,700	42,335	161,772	518,103	415,536	1,485,053

41. Double Bar Diagrams. In certain diagrams it is advantageous to have the bars extend in both directions from the base line as in the following figure which gives the distribution by age and sex of the total population for 1910.

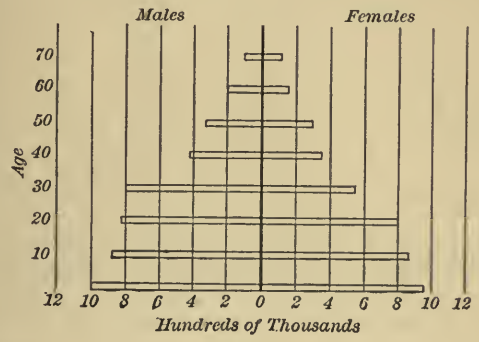


FIG. 7

EXERCISES

Make corresponding figures for the distribution by age periods and sex for 1910, in per cents, of

	1. Native Whites of Native Parentage.		2. Negroes.	
Age.	Male.	Female.	Male.	Female.
Under 5	6.7	6.5	6.4	6.5
5– 9	6.0	5.8	6.3	6.4
10–14	5.5	5.3	5.9	5.9
15–19	5.2	5.1	5.2	5.6
20–24	4.7	4.7	4.9	5.6
25–29	4.1	4.0	4.3	4.7
30–34	3.5	3.4	3.4	3.4
35–39	3.2	3.0	3.3	3.2
40–44	2.6	2.4	2.3	2.3
45–49	2.2	2.0	2.0	1.9
50–54	2.1	1.8	1.8	1.5
55–59	1.6	1.4	1.2	1.0
60–64	1.3	1.2	1.0	0.9
65–69	1.0	0.9	0.7	0.6
70–74	0.6	0.6	0.4	0.4

3. Make a diagram displaying the following data on the average yields and values per acre of Iowa farm crops for 1909.

Crop.	Yield.	Value.
Corn	37.1 bu.	$18.60
Oats	27.5	10.54
Wheat	15.3	14.62
Barley	19.2	9.31
Rye	13.6	8.50
Flaxseed	9.1	11.74
Timothy seed	4.2	5.79
Hay and forage	32.0 cwt.	11.76
Potatoes	86.8 bu.	39.10

4. Make a diagram showing the weight in pounds and value of the dairy products shipped from Humboldt County, California, in 1913.

Article.	Weight in Lbs.	Value.
Butter	5,793,620	$1,796,190
Cheese	304,570	54,820
Condensed milk	1,302,560	112,720
Dry milk	1,692,100	157,430
Fresh cream and buttermilk	277,800	6,920
Casein	1,484,910	89,100

42. Dot Diagrams. The following diagram taken from the U. S. census reports gives the number of all sheep on farms April 15, 1910.

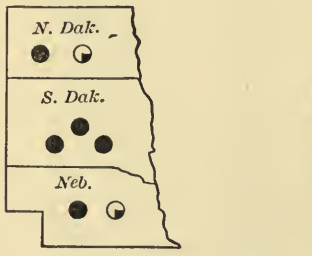


Fig. 8

EXERCISES

1. Make a corresponding chart showing all sheep on farms April 15, 1910 for

Wyoming	5,397,000	Utah	1,827,000
Montana	5,380,000	Colorado	1,400,000
Idaho	3,010,000	Nevada	1,150,000

2. Make a dot diagram showing all fowls on farms in the states given on April 15, 1910. [Here it is convenient to let • stand for 1,000,000.]

North Dakota	3,268,000	Iowa	23,482,000
South Dakota	5,251,000	Minnesota	10,697,000
Nebraska	9,351,000	Montana	966,000

43. Circular Diagrams. The following diagram shows the relative percentage of improved and unimproved land area in farms for the total land area of the U. S. 1850–1880–1910. (U. S. census report 1910.)

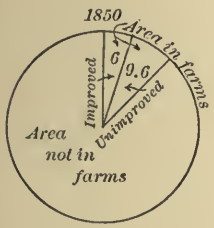


Fig. 9

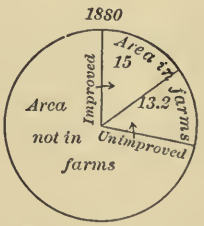


Fig. 10

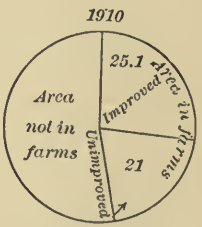


Fig. 11

The circles indicate by the size of their sectors the relative ratio of lands improved and unimproved in farms to the total land area of the U. S. Note the rapid decrease in the area not in farms, also the increase in the proportion of improved to the unimproved. In 1910 less than 50% of the total area is in farms.

EXERCISES

1. Make a circular diagram showing in percents the relative importance of the several countries in the production and consumption of cotton.

United States	60.9	Russia	4.5
India	17.1	Brazil	1.9
Egypt	6.6	All others	3.6
China	5.4		

2. Make circular diagrams showing per cent. distribution of foreign born population by principal countries of birth for the years indicated.

	1850.	1870.	1890.	1910.
Germany	26.0	30.4	30.1	18.5
Ireland	42.8	33.3	20.2	10.0
Canada and New Foundland	6.6	8.9	10.6	9.0
Great Britain	16.9	13.8	13.5	9.0
Norway, Sweden and Denmark	0.8	4.3	10.1	9.3
Austria-Hungary		1.3	3.3	12.4
Russia and Finland	0.1	0.1	2.0	12.8
Italy	0.2	0.3	2.0	9.9
All others	6.6	7.6	8.2	9.1

3. Make circular diagrams for the years 1900 and 1910 showing per cent. of total value of farm property represented by the items mentioned.

	1910.	1900.
Land	69.5	63.9
Buildings	15.4	17.4
Implements and machinery	3.1	3.7
Domestic animals, poultry, and bees	12.0	15.0

(Compare with Ex. 4, p. 48.)

44. Different Shadings or Colors are sometimes used in maps to represent different statistical facts. The annexed chart gives the average value of farm land per acre in Delaware. The average for the state is $33.63.

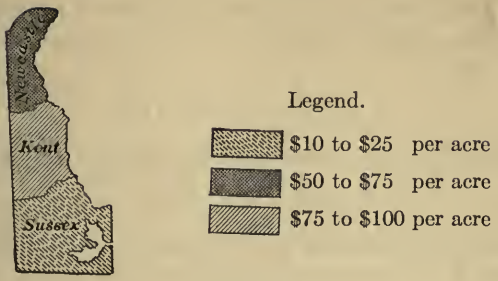


FIG. 12

EXERCISES

1. Draw a map of Connecticut showing the counties and mark to show the average value of farm land per acre. Average value for state is $33.03. Average value by counties is: Fairfield $75 to $100 per acre. New Haven and Hartford $25 to $50 per acre. Litchfield, Tolland, Windham, Middlesex, and New London $10 to $25 per acre.

2. Draw a map, give legend, and mark to show per cent. of improved land in farms operated by tenants by states in 1910. Utah, less than 10 per cent. Wyoming, 10 to 20 per cent. Colorado and Missouri, 20 to 30 per cent. Kansas, Nebraska, and Iowa, 30 to 40 per cent. Illinois, 40 to 50 per cent.

45. **Distance between two Points.** Let P_1 and P_2 be the end points of a given segment in the plane. P_1 and P_2 are given points, *i.e.*, their coördinates (X_1, Y_1) and (X_2, Y_2) are given or known numbers.

We wish to find the length of the segment P_1P_2 in terms of x_1, y_1, x_2, y_2; or, in other words to find the distance between two given points.

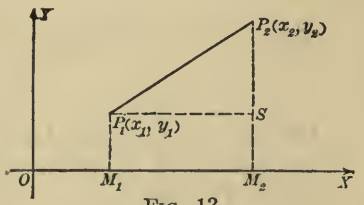


FIG. 13

Through P_1 draw a line parallel to the x-axis, and through P_2 a line parallel to the y-axis intersecting the first

in S. Then whatever the relative positions of P_1 and P_2 in the plane, the measure of P_1S is $x_2 - x_1$, and the measure of SP_2 is $y_2 - y_1$; also

$$\overline{P_1P_2}^2 = \overline{P_1S}^2 + \overline{SP_2}^2.$$

Therefore if we let d represent the required distance P_1P_2,

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \tag{1}$$

It is clearly immaterial which of the two points is called P_1 and which P_2, so the formula may also be written in the equivalent form

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}, \tag{1}$$

and may be expressed in words thus: *The distance between two points given by their rectangular coördinates is equal to the square root of the sum of the square of the difference of the abscissas and the square of the difference of the ordinates.*

EXAMPLE. The distance from the point $(2, -7)$ to the point $(7, 5)$ is

$$d = \sqrt{5^2 + 12^2} = 13.$$

46. Ratio of Division. Let P_1 and P_2 be two fixed points on a line and P any third point. Then the point P is said to divide the segment P_1P_2 in the ratio

$$\frac{P_1P}{P_1P_2} = \lambda. \tag{2}$$

This ratio λ is called the ***ratio of division*** or the ***division ratio.***

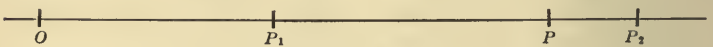


FIG. 14

If we choose an origin O on the given line then the abscissas x_1 of P_1 and x_2 of P_2 are known. Let us denote the abscissa of P by x. Then we have

$$P_1P = x - x_1, \qquad P_1P_2 = x_2 - x_1;$$

hence the abscissa x of P must satisfy the condition

$$\frac{x - x_1}{x_2 - x_1} = \lambda, \tag{3}$$

whence solving for x,

$$x = x_1 + \lambda(x_2 - x_1). \tag{4}$$

If the segments P_1P and P_1P_2 have the same sense, the division ratio is positive and P and P_2 lie on the same side of P_1. If the segments P_1P and P_1P_2 are oppositely directed, then the division ratio is negative and P and P_2 are on opposite sides of P_1. Thus, if the abscissas of P_1 and P_2 are 2 and 14, the abscissas of the points that divide P_1P_2 in the ratios $\frac{1}{3}$, $\frac{1}{2}$, $\frac{2}{3}$, $-\frac{1}{2}$, -1, -2 are 6, 8, 10, -4, -10, -22.

47. Point of Division. *To find the coördinates of the point which divides the line joining two given points in a given ratio* λ.

Let $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ be the two given points, λ the given ratio, and $P(x, y)$ the required point.

Draw P_1Q_1, PQ, P_2Q_2 parallel to the y-axis, and P_1R_1, PR, P_2R_2 parallel to the x-axis. Then Q and R divide Q_1Q_2 and R_1R_2, respectively, in the ratio λ. Now as $OQ_1 = x_1$, $OQ_2 = x_2$, $OQ = x$, it follows from (4) § 46 that

$$x = x_1 + \lambda(x_2 - x_1). \tag{5}$$

In the same way we find

$$y = y_1 + \lambda(y_2 - y_1). \tag{6}$$

Thus, the coördinates x, y of P are expressed in terms of the coördinates of P_1 and P_2 and the division ratio λ.

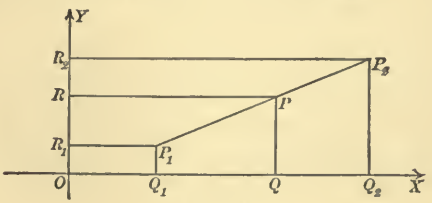


FIG. 15

48. Middle Point. If P be the middle point of P_1P_2, $\lambda = \frac{1}{2}$, and

$$x = \tfrac{1}{2}(x_1 + x_2), \qquad y = \tfrac{1}{2}(y_1 + y_2).$$

That is, *the abscissa of the mid-point of a segment is one half the sum of the abscissas of its end points, and the ordinate is one half the sum of the ordinates.*

EXERCISES

1. Find the lengths of the sides of the following triangles:

(*a*) (4, 8), (− 4, − 8), (1, 4). (*b*) (4, 5), (4, − 5), (− 4, 5).
(*c*) (2, 1), (− 1, 2), (− 3, 0). (*d*) (− 2, 1), (− 3, − 4), (2, 0).
(*e*) (2, 3), (1, − 2), (3, 8). (*f*) (5, 2), (− 3, — 2), (7, 3).

What inference can be drawn from the answers to (*e*) and (*f*)?

2. Find the lengths of the sides and of the diagonals of the quadrilateral (2, 1), (5, 4), (4, 7), (1, 4).

3. A (0,2), B (3, 0), and C (4, 8) are the vertices of a triangle. Show that the distance from A to the mid-point of BC is one-half the length of BC.

4. Show that two medians of the triangle (1, 2), (5, 5), (− 2, 6) are equal. What inference can you draw?

5. The ends of one diagonal of a parallelogram are (4, − 2) and (− 4, − 4). One end of the other diagonal is (1, 2). Find the other end.

6. The end points of a segment PQ are (1, − 3) and (5, 0). Find the length of the segment, and the lengths of its projections on the x and y axes.

7. Show that (0, 10), (1, 1), (5, 6) are the vertices of an isosceles right triangle.

8. Find the coördinates of the point

(*a*) Two-thirds of the way from (− 1, 7) to (8, 1);
(*b*) Two-thirds of the way from (8, 1) to (− 1, 7);
(*c*) Four-sevenths of the way from (1, − 7) to (8, 0);
(*d*) Three-sevenths of the way from (8, 0) to (1, − 7).

9. The segment from (4, 5) to (2, 3) is produced half its length. Find the end point.

49. Locus of a Point in a Fixed Plane. If a point is forced to move so as to be always equidistant from two fixed points, we know that it must lie on the perpendicular bisector of the segment joining these points. If a point must be at a constant distance from a fixed point, it will lie on a circle. If a point must be always equidistant from a fixed point and a fixed line, it will lie on a certain curve, called a *parabola*, which we have not yet studied.

If x and y are the coördinates of a point P, the values of x and y change as P moves in the plane. For this reason they are called *variables*. If P is subject to a condition which forces it to lie on a certain curve, then x and y must satisfy a certain condition which can be expressed as an equation in x and y.

For example, if P is always equidistant from (1, 2) and (2, 1), then, for all positions of P, $x - y = 0$. If P is always equidistant from (0, 2) and the x-axis then $x^2 - 4y + 4 = 0$. If P is always 3 units from the origin, then $x^2 + y^2 = 9$.

Whenever a plane curve and an equation in x and y are so related that every point on the curve has coördinates which satisfy the equation, and conversely, every real solution of the equation furnishes coördinates of a point on the curve, then the equation is called the ***equation of the curve***, and the curve is called the ***locus of the equation***. This dual relation between equation and curve is the subject of study in Analytic Geometry.

50. Equation of a Locus. To find the equation of the locus of a point which moves in a plane according to some stated law, we proceed as follows: *First*, draw a pair of coördinate axes; and locate and denote by appropriate numbers or letters all fixed distances, including the coördinates of fixed points. *Second*, mark a point P with coördinates x and y, to represent the moving point; express the conditions of the problem in terms of x, y, and the given constants; and simplify the resulting equation.

Third, show that every real solution of the equation so obtained gives a point which satisfies the conditions governing the motion of P.

EXAMPLE. Find the equation of the locus of a point which is always equidistant from a fixed line and a fixed point.

First. We are free to choose the axes where we please. It is convenient to take the fixed line for the x-axis, and to take the y-axis through the fixed point. Then the coördinates of the fixed point may be called $(0, a)$.

Second. The distance from $P(x, y)$ to the fixed line is y, and its distance to the fixed point $(0, a)$ is $\sqrt{x^2 + (y - a)^2}$. Hence the condition expressed in the problem gives $y = \sqrt{x^2 + (y - a)^2}$. This simplifies to

$$x^2 + 2ay = a^2.$$

Third. It is easy to show, by reversing the above process, that if $x = h$, $y = k$, is any solution of this equation, then the point $Q\ (h, k)$ is equidistant from the x-axis and the point $(0, a)$.

Therefore $x^2 + 2ay = a^2$ is the required equation.

EXERCISES

1. Find the equation of the locus of a point which moves so that:

(a) it is equidistant from the coördinate axes;

(b) it is four times as far from the x-axis as from the y-axis;

(c) the sum of its distances from the axes is 6;

(d) the square of its distance from the x-axis is four times its distance from the y-axis.

2. Find the equation of the locus of a point that is always equidistant from $(4, -2)$ and $(7, 3)$. *Ans.* $3x + 5y = 19$.

3. Find the equation of the perpendicular bisector of the segment joining the two points (a, b) and (c, d).

Ans. $(a - c)x + (b - d)y = \frac{1}{2}(a^2 + b^2 - c^2 - d^2)$.

4. Find the equation of the locus of a point whose distance from the point $(-3, 4)$ is always equal to 5. *Ans.* $x^2 + y^2 + 6x - 8y = 0$.

5. Find the equation of the circle whose center is (a, b) and whose radius is c.

51. Locus of an Equation. In general a single equation in x and y has an infinite number of real solutions. Each of these solutions furnishes the coördinates of a point on the locus.

To find solutions and plot points on the curve, solve the equation, if possible, for y in terms of x, or *vice versa*. Determine and tabulate a convenient number of solutions by assigning values to x and computing the corresponding values of y. Using these for coördinates, plot the points which they represent and draw a smooth curve through the plotted points.

EXAMPLE 1. Construct the locus of the equation

$$x^2 = 4(x + y).$$

Solving the given equation for y we have

$$y = \frac{x^2}{4} - x.$$

Assigning to x the values 0, 1, 2, 3, etc., -1, -2, -3, etc., and computing the corresponding values of y, we have the following solutions.

$x\ldots$	0	1	2	3	4	5	6	7	-1	-2	-3
$y\ldots$	0	$-.75$	-1	$-.75$	0	1.25	3	5.25	1.25	3	5.25

We choose the axes, as in Fig. 16, so that all these points will go on the sheet.

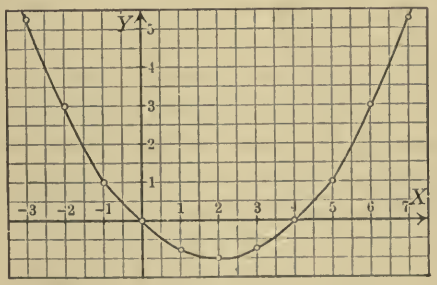


FIG. 16

On plotting the points and drawing a smooth curve through them, we have a sketch of the locus as shown.

EXAMPLE 2. Plot the curve whose equation is

$$x^2 + y^2 = 6x + 2y.$$

Solving the given equation for y, we have

$$y = 1 \pm \sqrt{1 + 6x - x^2},$$

and we tabulate solutions as follows.

x....	0	1	2	3	4	5	6	7	-1	-2
y....	0	-1.45	-2	-2.16	-2	-1.45	0	imag.	imag.	imag.
	2	3.45	4	4.16	4	3.45	2			

We note that each value of x gives two values of y, *i.e.* there are two points on the curve having the same abscissa. We find also that values of $x \geqq 7$ do not give real values of y and that the same is true for values of $x \leqq -1$.

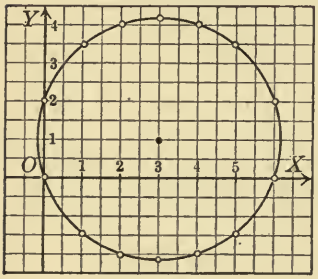


FIG. 16a

When these points have been plotted and a curve drawn through them we have the locus as shown in Fig. 16a.

52. Study of the Equation. Important facts about the shape and extent of the locus can be learned by a study of its equation. In the first example above, the equation is of the first degree in y. From this we infer that every value of x, without exception, gives exactly one value of y. Therefore every vertical line cuts the curve in one and only one point. As x increases beyond 2, y always increases, and the curve goes off beyond all limit in the first quadrant. The same is true in the second quadrant. On the other hand, the equation is of the second degree in x. When solved, it gives

$$x = 2 \pm 2\sqrt{1 + y};$$

hence every value of y greater than -1 gives two real values of x but every value of y less than -1 gives an imaginary value of x. Hence every horizontal line above $y = -1$ cuts the curve in two points, but there are no points on the curve below $y = -1$.

The equation of the second example, when solved for y as above, shows that values of x which make $1 + 6x - x^2 < 0$ give imaginary values for y. Hence there are no points on the curve to the left of the line $x = 3 - \sqrt{10} = -0.16$, nor to the right of the line $x = 3 + \sqrt{10} = 6.16$, but every vertical line between these limits cuts the curve in two points.

If we solve the same equation for x, we find

$$x = 3 \pm \sqrt{9 + 2y - y^2};$$

hence there are no points below the line $y = 1 - \sqrt{10} = -2.16$ nor above the line $y = 1 + \sqrt{10} = 4.16$, but every horizontal line between these lines cuts the curve in two points.

If the equation is a polynomial in x and y equated to zero, a glance will show whether or not it passes through the origin.

The intercepts * can be found by the rule: *To find the x-intercepts let* $y = 0$ *and solve for* x. Similarly find the y-intercepts.

53. **Symmetry.** Two points A and B are said to be ***symmetric with respect to a point*** P when the line AB is bisected by P.

Two points A and B are said to be ***symmetric with respect to an axis*** when the line AB is bisected at right angles by the axis.

If the points of a curve can be arranged in pairs which are symmetric with respect to an axis or a point, then the curve itself is said to be ***symmetric with respect to that axis or point.***

RULE I. *If the equation of a locus remains unchanged in form when in it* y *is replaced by* $-y$, *then the locus is symmetric with respect to the axis of* x.

For, if (x, y) can be replaced by $(x, -y)$ throughout the equation without affecting the locus, then if (a, b) is on the

* The intercepts of a curve on the axis of x are the abscissas of the points of intersection of the curve and the x-axis. The intercepts on the y-axis are the ordinates of the points of intersection of the curve and the y-axis.

locus, $(a, -b)$ is also on the locus, and the points of the locus occur in pairs symmetric with respect to the axis of x.

We can also prove the following rules.

Rule II. *If the equation of a locus remains unchanged in form when in it x is replaced by $-x$, then the locus is symmetric with respect to the y-axis.*

Rule III. *If the equation of a locus remains unchanged in form when in it x and y are replaced by $-x$ and $-y$, then the locus is symmetric with respect to the origin.*

54. Points of Intersection. If two curves whose equations are given intersect, the coördinates of each point of intersection must satisfy both equations when substituted in them for x and y. In algebra it is shown that *all* values satisfying two equations in two unknowns may be found by regarding these equations as simultaneous in the unknowns and solving. Hence the rule to find the points of intersection of two curves whose equations are given.

Consider the equations as simultaneous in the coördinates, and solve for x and y.

Arrange the real solutions in corresponding pairs. These will be the coördinates of all of the points of intersection.

EXERCISES

Plot the loci of the following equations:

1. $2x - 3y - 6 = 0$.
2. $4x - 6y - 6 = 0$.
3. $6x - 9y + 36 = 0$.
4. $2x + 3y + 5 = 0$.
5. $3x - 2y - 12 = 0$.
6. $5x + 2y - 4 = 0$.
7. $y = 7x - 3$.
8. $2y - x = 2$.
9. $2x + 9y + 13 = 0$.
10. $(x - 4)(y + 3) = 0$.
11. $(x^2 - 4)(y - 2) = 0$.
12. $4x^2 - y^2 = 0$.
13. $6x^2 + 5xy - 6y^2 = 0$.
14. $x^2 + y^2 = 4$.
15. $x^2 - y^2 = 4$.
16. $x^2 + y^2 = 25$.
17. $(x - 8)^2 + (y - 4)^2 = 25$.
18. $(x - 4)^2 + (y - 2)^2 = 5$.
19. $4(x + 1) = (y - 2)^2$.
20. $10y = (x + 1)^2$.
21. $y = x^3 - 4x^2 - 4x + 16$.

22. $y = x, x^2, x^3, x_4, \cdots, x^n$. What points are common to these curves?
23. $y^2 = x, x^2, x^3, x^4$.
24. $y = (x-1), (x-1)^2, (x-1)^3$.
25. $y = (x-1)(x-2)(x-3)$.
26. $y = (x-1)(x-2)^2$.
27. $y = (x-2)^3$.
28. $y^2 = (x-1)(x-2)(x-3)$.
29. $y^2 = (x-1)(x-2)^2$.
30. $y^2 = (x-2)^3$.
31. $y = \dfrac{x}{x-1}$.
32. $y = \dfrac{x}{x+1}$.
33. $y = \dfrac{x}{x^2+1}$.
34. $y = \dfrac{x^2}{x^2+1}$.
35. $y = \dfrac{(x-1)(x-3)}{(x-2)(x-4)}$.
36. $y = \dfrac{(x+1)(x-3)}{(x-2)(x+4)}$.

Find the points of intersection of the following curves:

37. $\begin{cases} 2x + y = 5, \\ x + 2y = 1. \end{cases}$
38. $\begin{cases} x - y = 2, \\ 2x - 3y = 1. \end{cases}$
39. $\begin{cases} x^2 + y^2 = 18, \\ 2x - y = 3. \end{cases}$

Ans. (3, 3), $(-\frac{3}{5}, -\frac{21}{5})$.

40. $\begin{cases} x^2 + y^2 = 18, \\ y^2 = 3x. \end{cases}$

Ans. (3, 3), (3, −3).

41. $\begin{cases} 3x^2 + 4y^2 = 48, \\ x - y + 1 = 0. \end{cases}$

Ans. (2, 3), $(-\frac{32}{7}, -\frac{15}{7})$.

42. $\begin{cases} 3x^2 - 4y^2 = 11, \\ 4x = 3y^2. \end{cases}$

Ans. (3, 2), (3, −2).

43. $\begin{cases} xy = 2, \\ y^2 = 4x. \end{cases}$
44. $\begin{cases} xy = 2, \\ x^2 + y^2 = 5. \end{cases}$
45. $\begin{cases} xy = x + y + 1, \\ y = x - 1. \end{cases}$

Ans. (3, 2).

46. $\begin{cases} xy = x + y + 1, \\ 4x - 3y + 1 = 0. \end{cases}$

Ans. (2, 3), $(-\frac{1}{2}, -\frac{1}{3})$.

47. Find the length of the common chord of the two circles $x^2 + y^2 = 4x$ and $x^2 + y^2 = 4(x + y - 1)$. *Ans.* $2\sqrt{3}$.

48. In what respects are the loci of the following equations symmetric?

(*a*) $y = x^2$.
(*b*) $y^2 = x$.
(*c*) $y = x^3$.
(*d*) $y^2 = x^3$.
(*e*) $y^3 = x^2$.
(*f*) $y^2 = x^4$.
(*g*) $y = x^3 - x$.
(*h*) $y = x^4 - x^2$.
(*i*) $x^3 - y^3 - x - y = 0$.
(*j*) $xy = a$.
(*k*) $ax^2 + by^2 = 1$.
(*l*) $ax^2 + 2bxy + cy^2 = 1$.

55. Straight Line Parallel to an Axis. Suppose a point moves about on a piece of coördinate paper in such a way that it is always two units to the right of the axis of y. It would

evidently be on the line AB that is parallel to the y-axis and at a distance of two units to the right of OY. Every point of the line AB has an abscissa of two ($x = 2$), and every point whose abscissa is two lies on the line AB. For this reason we *say* that the equation

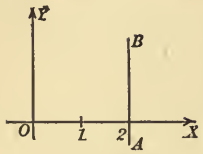


Fig. 17

$$x = 2$$

represents the line AB or is the *equation of the line AB.*

More generally, the equation

$$x = a,$$

where a is any real number, *represents a straight line parallel to the y-axis and at a distance a from it.* Similarly, the equation $y = b$ represents a line parallel to the x-axis.

56. Straight Line through the Origin. Suppose a point moves about on a piece of coördinate paper in such a way that its distance from the x-axis, represented by y, is always equal to m times its distance from the y-axis, represented by x. The equation of the locus of the point is

$$y = mx.$$

This is the equation of a straight line through the origin. The points of this line have the property that the ratio y/x of their coördinates is the same number m, wherever on this line the point is taken. Moreover for any point Q, not on this line, the ratio y/x must evidently be different from m. *The number m is called the* ***slope*** *of the line.*

57. Proportional Quantities. Whenever two quantities y and x vary in such a manner that their ratio y/x is always constant, say m, they are said to be *proportional.* The constant m is called the *factor of proportionality.* Many instances occur

in the applied sciences of two quantities related in this manner. It is often said that one quantity *varies* as the other. Thus *Hooke's law* states that the elongation E of a stretched wire or spring varies as the tension t; that is, $E = kt$, where k is a constant. For a given wire, when E was expressed in thousandths of an inch and t in pounds, the following relation was found:

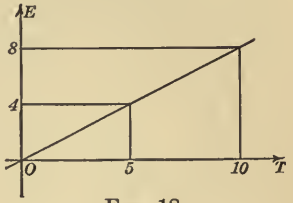


FIG. 18

$$E = .8t$$

Thus when $t = 10$, $E = 8$ and when $t = 5$, $E = 4$.

EXERCISES

Draw the lines

1. $x = 1, -1, 0, 2, 3, -2, -3, -4, 4.$
2. $y = 1, -1, 0, 2, 3, -2, -3, -4, 4.$
3. What is the locus of a point if $x > 3$? $x = 3$? $x < 3$?
4. What is the locus of a point if $2 < x < 3$? $2 \leq x < 3$? $2 \leq x \leq 3$? $2 < x \leq 3$?
5. What is the locus of a point if $2 < x < 3$ and $1 < y < 2$?
6. What is the locus of a point if $x^2 + y^2 < 16$ and $x > 2$?
7. What is the locus of a point if $9 < x^2 + y^2 < 16$?
8. A stand-pipe is filled at the rate of 150 gallons per hour. What is the amount A of water in the stand-pipe h hours after filling begins?
9. A man saves \$50 each month and deposits it in a bank. What is the amount A which he has in the bank after t months?
10. A railroad track has a rise of 1 ft. in 20. Give its equation and plot.
11. The extension E in feet of a spiral spring due to a tension t of 1 lb., was 1 inch. What is the relation connecting E and t? (Use Hooke's law.)

58. Slope of a Straight Line. The *slope*, m, of the line passing through two points $P_1(x_1, y_1)$, $P_2(x_2, y_2)$, Fig. 19, is given by the formula

$$m = \frac{RP_2}{P_1R} = \frac{y_2 - y_1}{x_2 - x_1}. \tag{8}$$

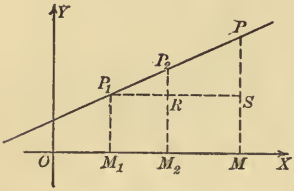


FIG. 19

59. Equation of a Line through two Points. Let the two given points be $P_1(x_1, y_1)$, $P_2(x_2, y_2)$. Let $P(x, y)$ be any other point on the line joining P_1P_2. Draw P_1RS parallel to the x-axis. Draw P_1M_1, P_2M_2, PM, parallel to the y-axis. Then since the triangles P_1SP and P_1RP_2 are similar, we have

$$\frac{SP}{P_1S} = \frac{RP_2}{P_1R},$$

or

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1},$$

which may be written in the form

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1). \tag{9}$$

The equation of a straight line with a given slope m and passing through a given point (x_1, y_1) is seen from the last equation to be

$$y - y_1 = m(x - x_1). \tag{10}$$

In particular if the y-intercept is given as b, the equation of the straight line having the given intercept and with slope m is

$$y - b = m(x - 0)$$

which reduces to

$$y = mx + b. \tag{11}$$

This last equation is called the ***slope form*** of the equation of the straight line.

If both intercepts are given, say x-intercept $= a$, y-intercept $= b$, we can find the equation of the line by means of the equation for a line through two given points. We have

$$y - b = \frac{0 - b}{a - 0}(x - 0),$$

which reduces to

$$\frac{x}{a} + \frac{y}{b} = 1. \tag{12}$$

This is called the ***intercept form*** of the equation of the straight line.

60. Parallel Lines. Consider two parallel lines P_1R_1 and P_2R_2. Draw R_2R_1 and P_2P_1 parallel to the y-axis, and R_1S_1, R_2S_2 parallel to the x-axis. Then since the triangles $R_1S_1P_1$ and $R_2S_2P_2$ are equal,

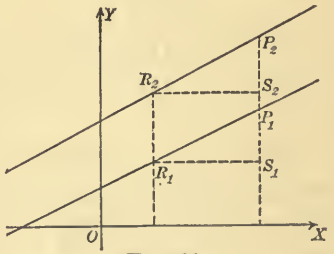


FIG. 20

$R_1S_1 = R_2S_2$ and $S_1P_1 = S_2P_2$. Hence,

$$\frac{S_2P_2}{R_2S_2} = \frac{S_1P_1}{R_1S_1}.$$

That is *the slopes of any two non-vertical parallel lines are equal.*

61. Perpendicular Lines. Consider two perpendicular lines L_1 and L_2 intersecting at $P_1(x_1, y_1)$. Let $P_2(x_1 + a, y_1 + b)$ be a second point on L_1; then since the given lines are perpendicular, the point $Q_2(x_1 - b, y_1 + a)$ lies on L_2 as shown by construction in the figure. Then the slope of L_1 is $m_1 = b/a$, by the definition of slope, § 58; and the slope of L_2 is $m_2 =$

$-(a/b)$, for the same reason. It follows that we have

$$m_1 m_2 = -1. \tag{13}$$

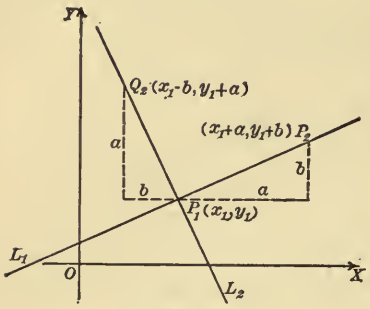


FIG. 21

This proves the theorem: *If two non-vertical lines are perpendicular, then the product of their slopes is* -1.

The converse is also true: *If the product of the slopes of two lines is* -1, *then they are perpendicular.* The proof, which is suggested by Fig. 21, is left to the student.

62. General Equation of the First Degree. The equation

$$Ax + By + C = 0, \tag{14}$$

where A, B, C are constants, is called the general equation of the first degree in x and y because every equation of the first degree may be reduced to that form. For any values whatsoever of A, B, and C, provided A and B are not both zero, the general equation of the first degree represents a straight line.

EXERCISES

1. Find the slope of the line joining the points

(*a*) (1, 3) and (2, 7). (*b*) (2, 7) and $(-4, -4)$.

(*c*) $(\sqrt{3}, \sqrt{2})$ and $(-\sqrt{2}, \sqrt{3})$. (*d*) $(a+b, c+a)$ and $(c+a, b+c)$.

2. Prove by means of slopes that $(-4, -2)$, (2, 0), (8, 6), (2, 4) are the vertices of a parallelogram.

3. Prove by means of slopes that (0, -2), (4, 2), (0, 6), $(-4, 2)$ are the vertices of a rectangle.

4. What are the equations of the sides of the figures in Exs. 2 and 3.

5. Find the intercepts and the slope of each of the following lines:

(*a*) $2x + 3y = 6$. (*b*) $x - 2y + 5 = 0$.
(*c*) $3x - y + 3 = 0$. (*d*) $5x + 2y - 6 = 0$.
(*e*) $7x - 4y - 28 = 0$. (*f*) $3y - 2x = 8$.

6. Find the equations of the lines satisfying the following conditions:

(*a*) passing through $(-3, 1)$ and slope $= 2$.
(*b*) having the x-intercept $= 3$, y-intercept $= -2$.
(*c*) slope $= -3$, x intercept $= 4$.
(*d*) x intercept $= -3$, y intercept $= -4$.
(*e*) passing through the point $(2, 3)$ and with slope $= -2$.

7. Find the points of intersection of

(*a*) $x - 7y + 25 = 0$, $x^2 + y^2 = 25$.
(*b*) $2x^2 + 3y^2 = 35$, $3x^2 - 4y = 0$.
(*c*) $x^2 + y = 7$, $y^2 - x = 7$.
(*d*) $y = x + 5$, $9x^2 + 16y^2 = 144$.

8. Find the equations, and reduce them to the general form, of the lines for which

(*a*) $m = 2$, $b = -3$. (*b*) $m = -1/2$, $b = 3/2$.
(*c*) $m = 2/5$, $b = -5/2$. (*d*) $m = 1$, $b = -2$.
(*e*) $a = 3$, $b = 3$. (*f*) $a = 4$, $b = 2$.
(*g*) $a = -3$, $b = -3$. (*h*) $a = 4$, $b = -2$.
(*i*) $a = -3$, $b = 3$. (*j*) $a = 2$, $b = 4$.

9. Write the equations of the lines passing through the points:

(*a*) $(-2, 3)$, $(-3, -1)$. (*b*) $(5, 2)$, $(-2, 4)$.
(*c*) $(1, 4)$, $(0, 0)$. (*d*) $(2, 0)$, $(-3, 0)$.
(*e*) $(0, 2)$, $(3, -1)$. (*f*) $(2, 3)$, $(-6, -5)$.

10. Write the equations of the lines passing through the given points and with the given slopes:

(*a*) $(-2, 3)$, $m = 2$. (*b*) $(5, 2)$, $m = 1$.
(*c*) $(1, 4)$, $m = \frac{1}{2}$. (*d*) $(2, 0)$, $m = -\frac{2}{3}$.
(*e*) $(0, 2)$, $m = 0$. (*f*) $(3, -2)$, $m = -2$.

11. Write the equation of the line which shall pass through the intersection of $2y + 2x + 2 = 0$ and $3y - x - 8 = 0$, and having a slope $= 4$. *Ans.* $16x - 4y + 51 = 0$.

12. What are the equations of the diagonals of the quadrilateral the equations of whose sides are $y - x + 1 = 0$, $y = -x + 2$, $y = 3x + 2$, and $y + 2x + 2 = 0$?

13. Required the equation of the line which passes through $(2, -1)$ and is

(*a*) parallel to $2y - 3x - 5 = 0$. *Ans.* $2y - 3x + 8 = 0$.

(*b*) perpendicular to $2y - 3x - 5 = 0$. *Ans.* $2x + 3y - 1 = 0$.

14. Find the equations of the two straight lines passing through the point (2, 3), the one parallel, the other perpendicular to the line $4x - 3y = 6$. *Ans.* $4x - 3y + 1 = 0$, $3x + 4y - 18 = 0$.

15. Passing through $(4, -2)$, the one parallel, the other perpendicular to the line $y = 2x + 4$. *Ans.* $y = 2x - 10$, $x + 2y = 0$.

16. Passing through the point of intersection of $4x + y + 5 = 0$ and $2x - 3y + 13 = 0$, one parallel, the other perpendicular to the line through the two points (3, 1) and $(-1, -2)$.

Ans. $3x - 4y + 18 = 0$, $4x + 3y - 1 = 0$.

17. Find the equation of the line joining the origin to the point of intersection of $2x + 5y - 4 = 0$ and $3x - 2y + 2 = 0$.

Ans. $y = -8x$.

18. Find the equation of the straight line passing through the point of intersection of $2x + 5y - 4 = 0$ and $2x - y + 1 = 0$ and perpendicular to the line $5x - 10y = 17$. *Ans.* $6x + 3y = 2$.

19. Find the equations of the lines satisfying the following conditions:

(*a*) through (2, 3), parallel to $y = 7x + 3$.

(*b*) through $(4, -1)$, perpendicular to $2x + 3y = 6$.

(*c*) through $(-2, -1)$, parallel to $3y - 2x = 1$.

(*d*) through $(3, -6)$, parallel to $2y + 4x = 7$.

(*e*) through $(-1, -1)$, perpendicular to $x/2 + y/3 = 1$.

(*f*) through (2, 2), perpendicular to $y = -3x + 2$.

20. Prove that the diagonals of a parallelogram bisect each other.

21. Prove that the diagonals of a rhombus bisect each other at right angles.

22. Prove that the diagonals of a square are equal and bisect each other at right angles.

23. A straight line makes an angle of 45° with the x-axis and its y intercept = 2; what is its equation? *Ans.* $y = x + 2$.

24. The following data gives the height of a plant in inches on different days.

Height.....	0	28	33	36	40	52	62	66
Day.....	0	40	60	80	100	120	140	160

Find the rate of growth after 60 days.

Find the rate of growth after 110 days.

[The rate of growth is the slope of the curve. The slope of a curve at a given point is defined to be the slope of the tangent line drawn to the curve at the given point. Draw the tangent with a ruler and with the aid of the eye.] *Ans.* 7/10 in. per day; 0.55 in. per day.

CHAPTER IV

LOGARITHMS

63. Definitions and Preliminary Notions. In the equation

$$10^2 = 100,$$

three numbers are involved. By omitting each number in turn there arise three different problems. If we omit the 100, we have the familiar question in involution:

$$10^2 = ?.$$

If we omit the 10 we have the familiar question in evolution:

$$?^2 = 100,$$

or, as it is usually written,

$$\sqrt{100} = ?.$$

If we omit the 2 we have the following question

$$10^? = 100,$$

which we agree to write in the form,

$$\log_{10} 100 = ?$$

and we say that 2 = *the* ***logarithm*** *of* 100 *to the* ***base*** 10.

In general, if

(1) $$b^x = N,$$

then x = *the* ***logarithm*** *of* N *to the* ***base*** b, and we write,

(2) $$x = \log_b N.$$

(1) and (2) are then simply two different ways of expressing the same relation between b, x, and N. (1) is called the *ex-*

ponential form. (2) is called the *logarithmic form.* Either of the statements (1) or (2), implies the other. The *exponent* in (1) is the *logarithm* in (2), a fact which may be emphasized by writing

(3) $$(\text{base})^{\text{logarithm}} = \text{number}.$$

For example, the following relations in exponential form:

$$3^2 = 9, \quad 2^4 = 16, \quad (1/2)^3 = 1/8, \quad a^y = x,$$

are written respectively in the logarithmic form:

$$2 = \log_3 9, \quad 4 = \log_2 16, \quad 3 = \log_{1/2} 1/8, \quad y = \log_a x.$$

We shall now give the following

DEFINITION OF A LOGARITHM. *The power to which a given number called the base must be raised to equal a second number is called the logarithm of the second number.*

EXERCISES

1. Write the following equations in logarithmic form:

(*a*) $9 = 3^2$. (*g*) $7 = 7^1$.
(*b*) $64 = 4^3$. (*h*) $25 = (\sqrt{5})^4$.
(*c*) $16 = 2^4$. (*i*) $8 = (\sqrt{2})^6$.
(*d*) $243 = 3^5$. (*j*) $3 = (\sqrt{3})^2$.
(*e*) $64 = 2^6$. (*k*) $3 = \sqrt{9}$.
(*f*) $2401 = 7^4$. (*l*) $4 = \sqrt[3]{64}$.

2. Write the following equations in exponential form:

(*a*) $\log_2 16 = 4$. (*g*) $\log_{10} 0.1 = -1$.
(*b*) $\log_4 16 = 2$. (*h*) $\log_2 1/4 = -2$.
(*c*) $\log_{10} 1000 = 3$. (*i*) $\log_{64} 2 = 1/6$.
(*d*) $\log_3 729 = 5$. (*j*) $\log_2 1/8 = -3$.
(*e*) $\log_5 625 = 4$. (*k*) $\log_{17} 1 = 0$.
(*f*) $\log_{12} 1728 = 3$. (*l*) $\log_a a = 1$.

3. Find the numerical value of each of th following:

(*a*) $\log_2 64$. (*e*) $\log_{25} 5$.
(*b*) $\log_{10} 0.001$. (*f*) $3 \log_5 625 + \log_2 16$.
(*c*) $\log_{27} 3$. (*g*) $\log_{1/2} 4$.
(*d*) $\log_{10} 100 - \frac{1}{2} \log_{0.1} 100$. (*h*) $5 \log_2 16 - 2 \log_{25} 625$.

64. Properties of Logarithms. Any positive number, except 1, may be the base of a system of logarithms of all the real positive numbers. In any such system,

1) *The logarithm of* 1 *is zero.*

For, $b^0 = 1$, therefore $\log_b 1 = 0$.

2) *The logarithm of the base itself is* 1.

For, $b^1 = b$, therefore $\log_b b = 1$.

3) *The logarithm of a product is the sum of the logarithms of the factors.*

For if $\log_b M = k$ and $\log_b N = l$, then $M = b^k$ and $N = b^l$, $MN = b^k \cdot b^l = b^{k+l}$, whence

$$\log_b MN = k + l = \log_b M + \log_b N.$$

This can readily be extended to three or more factors.

4) *The logarithm of a quotient is equal to the logarithm of the dividend minus the logarithm of the divisor.*

For,

$$\frac{M}{N} = \frac{b^k}{b^l} = b^{k-l},$$

therefore

$$\log_b \frac{M}{N} = k - l = \log_b M - \log_b N.$$

5) *The logarithm of the reciprocal of a number is the negative of the logarithm of the number.*

For on putting $M = 1$ under (4) above, we have

$$\log_b \frac{1}{N} = \log_b 1 - \log_b N = -\log_b N,$$

since $\log_b 1 = 0$.

6) *The logarithm of the pth power of a number is found by multiplying the logarithm of the number by p.*

For, $N = b^k$ and $N^p = (b^k)^p = b^{pk}$, whence

$$\log_b N^p = pk = p \log_b N.$$

7) *The logarithm of the rth root of a number is found by dividing the logarithm of the number by r.*

For, $N = b^k$ and $\sqrt[r]{N} = N^{1/r} = (b^k)^{1/r} = b^{k/r}$, whence

$$\log_b \sqrt[r]{N} = \frac{k}{r} = \frac{\log_b N}{r}.$$

EXERCISES

Express the logarithms of the following numbers in terms of the logarithms of integers. In this book, when the base is omitted, 10 is to be understood as the base.

1. $\log \dfrac{35^{2/3}}{13^{2/3} \cdot 6^{1/2}}$.

2. $\log \dfrac{17^{1/4}}{\sqrt[5]{10}\sqrt[7]{18}}$.

3. $\log \dfrac{12^{-2}}{2^{1/2}3^{1/3}}$.

4. Prove that $\log_3 \sqrt[3]{81\sqrt[4]{729 \cdot 9^{-2/3}}} = 31/18$.

Express the logarithms of the following in terms of the logarithms of prime numbers.

5. $\log \dfrac{(63)^{1/4}}{(25)^2(72)^{1/4}}$.

6. $\log \dfrac{88^{-1/2}}{(75)^{3/4}(12)^2}$.

7. $\log \dfrac{100^2}{20^{1/2}75^{2/3}}$.

8. $\log (\sqrt{2}\sqrt[3]{7^2}\sqrt[5]{6})$.

9. Given $\log 2 = 0.3010$, $\log 3 = 0.4771$, $\log 7 = 0.8451$, find the logarithms of the following numbers.

(*a*) 6.	(*e*) 32.	(*i*) 420.	(*m*) $\sqrt{1/2}$.
(*b*) 14.	(*f*) 10.5.	(*j*) 900.	(*n*) $\sqrt{504}$.
(*c*) 24.	(*g*) 14⅔.	(*k*) 35/48.	(*o*) $\sqrt[5]{13.5}$.
(*d*) 28.	(*h*) 2.52.	(*l*) 1/36.	(*p*) $\sqrt[3]{294}$.

10. Express the logarithms of each of the following expressions to the base a in terms of $\log_a b$, $\log_a c$, $\log_a d$.

(*a*) $b^{2/3}c^{-1/2}/d^{4/5}$. (*b*) $\sqrt[3]{a^{-2}\sqrt{b^6}} \div \sqrt{b^3\sqrt{a^{-3}}}$.

11. Prove that

$$\log_a \frac{x + \sqrt{x^2 - 1}}{x - \sqrt{x^2 - 1}} = 2 \log_a (x + \sqrt{x^2 - 1}).$$

12. If $\log 3 = 0.4771$, what is the (*a*) log 30? (*b*) log 300? (*c*) log 3000? (*d*) log 30,000? What part of these logarithms is the same? Why?

65. Computation of Common Logarithms. While any positive number except unity could be used as the base of a system of logarithms, only two systems are in general use. One, called the ***natural***, or ***Napierian system*** is used in analytical work and has the number $e = 2.71828 +$ for its base. The other, known as the ***common***, or ***Briggs system*** is used for all purposes involving merely numerical computations and has for its base the number 10. Unless specifically stated to the contrary the common system will be the one used throughout this book.

In the following discussion of common logarithms, $\log x$ is written as an abbreviation of $\log_{10} x$.

Every positive number has a common logarithm, and the value of this logarithm may be obtained correct to as many places of decimals as may be desired. Negative numbers and zero have no real logarithms.

If we extract the square root of 10, the square root of the result thus obtained, and so on, continuing the reckoning in each case to the fifth decimal figure, we obtain the following table:

$$\begin{aligned}
10^{1/2} &= 3.16228, & 10^{1/128} &= 1.01815,\\
10^{1/4} &= 1.77828, & 10^{1/256} &= 1.00904,\\
10^{1/8} &= 1.33352, & 10^{1/512} &= 1.00451,\\
10^{1/16} &= 1.15478, & 10^{1/1024} &= 1.00225,\\
10^{1/32} &= 1.07461, & 10^{1/2048} &= 1.00112,\\
10^{1/64} &= 1.03663, & 10^{1/4096} &= 1.00056,\\
&\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}$$

and so on. The exponents $\frac{1}{2}, \frac{1}{4}, \cdots$ on the left are the logarithms of the corresponding numbers on the right.

By the aid of this table we may compute the common logarithm of any number between 1 and 10, and hence of any positive number.

EXAMPLE. Find the common logarithm of 4.26.

Divide 4.26 by the next smaller number in the table, 3.16228. The quotient is 1.34719. Hence $4.26 = 3.16228 \times 1.34719$. Divide 1.34719 by the next smaller number in the table, 1.33352. The quotient is 1 0102. Hence $4.26 = 3.16228 \times 1.33352 \times 1.0102$. Continue thus, always dividing the quotient last obtained by the next smaller number in the table. We shall obtain by this method an expression for 4.26 in the form of a product:

$$\begin{aligned} 4.26 &= 3.16228 \times 1.33352 \times 1.00904 \times \cdots \\ &= 10^{1/2} \times 10^{1/8} \times 10^{1/256} \times \cdots \\ &= 10^{1/2+1/8+1/256+\cdots}. \end{aligned}$$

Therefore,

$$\begin{aligned} \log 4.26 &= \frac{1}{2} + \frac{1}{8} + \frac{1}{256} + \cdots \\ &= \frac{1}{2} + \frac{1}{8} + \frac{1}{256}, \text{ approximately} \\ &= .5000 \\ &\quad + .1250 \\ &\quad + .0039 \\ &\quad \overline{} \\ &= .6289 \end{aligned}$$

By referring to the table of logarithms at the end of the book we find that, correct to four decimal places,

$$\log 4.26 = .6294$$

Hence, by using only three terms in the above approximation we obtain a result which is in error but 5 units in the fourth decimal place.

66. Characteristic and Mantissa. If two numbers are unequal, their logarithms are unequal in the same sense; that is if

$$a < b < c,$$

then

$$\log a < \log b < \log c.$$

For example

$$\log 100 < \log 426 < \log 1000,$$

that is,

$$2 < \log 426 < 3.$$

When the logarithm of a number is not an integer it may be represented approximately by a decimal fraction correct to any desired number of places; thus $\log 426 = 2.6294$ to four decimal places.

The integral part of the logarithm is called the ***characteristic*** and the decimal part is called the ***mantissa.*** In $\log 426$, the characteristic is 2 and the mantissa is .6294. For convenience in computing it is desirable to have the mantissa positive even when the logarithm is a negative number. For example, $\log \frac{1}{2} = -0.3010$, but $-0.3010 = 9.6990 - 10$, and we write

$$\log \tfrac{1}{2} = 9.6990 - 10,$$

in which the characteristic is $9 - 10 = -1$, but the mantissa .6990 is positive.

It is convenient to write the logarithm of any number N in the form

$$\log N = M - k \cdot 10,$$

in which M is a positive number or zero and k is a positive integer or zero.

For example, $\log 426 = 2.6294$, $\log 42.6 = 1.6294$, $\log 4.26 = 0.6294$, $\log 0.426 = 9.6294 - 10$, $\log 0.0426 = 8.6294 - 10$, $\log 0.00426 = 7.6294 - 10$.

Moving the decimal point n places to the right (left) in a number increases (decreases) the characteristic of its common logarithm by n, but does not affect its mantissa.

For this has the effect of multiplying (dividing) the number by 10^n, and

$$\log (N \cdot 10^n) = \log N + \log 10^n = \log N + n$$

and

$$\log (N \div 10^n) = \log N - n.$$

Therefore, *the mantissa of the common logarithm of a number is independent of the position of the decimal point.* In other

words, the common logarithms of two numbers which contain the same sequence of figures differ only in their characteristics. Hence, tables of logarithms of numbers contain only the mantissas and the computer must determine the characteristics mentally. This can be done by the following simple rules.

RULE I. *The characteristic of the common logarithm of any number greater than* 1, *is one less than the number of digits before the decimal point.*

For if N is a number having n digits in the integral part (i. e. before the decimal point), then

$$10^{n-1} \leq N < 10^n$$

and

$$n - 1 \leq \log N < n;$$

therefore $\log N = (n - 1) +$ (a decimal fraction) and its characteristic is $n - 1$.

On the other hand if N is a decimal fraction (i. e., a positive number less than 1), we may move the decimal point 10 places to the right and apply Rule I., provided we subtract 10 from the resulting logarithm. For example,

$$\log 0.0006958 = \log 6958000 - 10$$

and by Rule I. the characteristic is $6 - 10$.

This process is easily seen to be equivalent to that specified in

RULE II. *To find the characteristic of the common logarithm of a number less than* 1, *subtract from* 9 *the number of ciphers between the decimal point and the first significant figure. From the number so obtained subtract* 10.

A very large number such as the distance in feet from the earth to the sun, 490,000,000,000 (correct to two significant figures), is conveniently written (on moving the decimal point 11 places to the left) in the form

$$4.9 \times 10^{11}$$

and the characteristic of its common logarithm is 11. Similarly a very small number such as 0.000,000,453,8 can be written (on moving the decimal point 7 places to the right),

$$4.538 \times 10^{-7}$$

and the characteristic of its logarithm is $-7 = 3 - 10$.

This form of expression is frequently used where only a few significant figures are known to be correct, and if the decimal point is placed *after the first significant figure*, the exponent of 10 is the characteristic of the logarithm of the number.

EXERCISES

Find the characteristics of the logarithms of the following numbers:

(1) 276.35
(2) 0.0495
(3) 1.837
(4) 6.3×10^{8}.
(5) 0.00072
(6) 4589.5
(7) 0.9372
(8) 7.32×10^{-5}.
(9) 73.187
(10) 8.421×10^{-26}.
(11) 7.268×10^{15}.
(12) 0.00008

67. Use of Tables. 1) *The characteristic* is not given in the table of logarithms. It is to be found by the above two rules. It should be written down first, and always expressed even though it be zero, in order to avoid error due to forgetting it.

2) *The mantissa* of the common logarithms of numbers, correct to four decimal places, are printed in Table I., at the end of the book. For convenience in printing the decimal points are omitted.

To find the mantissa of a number consisting of one, two, or three digits (exclusive of ciphers at the beginning or end, and the decimal point), look in the column marked N for the first two digits and select the column headed by the third digit; the mantissa will be found at the intersection of this row and this column. For example, to find the mantissa of 456, we run down the column headed N to 45 and then run across the page

to the column headed 6 where we find the mantissa .6590; again, the mantissa of 720 is found opposite 72 in the column headed 0, and is .8573.

EXERCISES

Look up the following logarithms in Table I.

(1) $\log 276 = 2.4409$
(2) $\log 8.64 = 0.9365$
(3) $\log .829 = 9.9186 - 10.$
(4) $\log 7.34 \times 10^{5} = 5.8657$
(5) $\log 2.30 \times 10^{-3} = 7.3617 - 10.$
(6) $\log 24700 = 4.3927$
(7) $\log 3.7 \times 10^{12}.$
(8) $\log 9.$
(9) $\log 846000.$
(10) $\log .000172$
(11) $\log .00782$
(12) $\log .0856$
(13) $\log 20.$
(14) $\log 8.5$
(15) $\log 1870.$
(16) $\log 3.20 \times 10^{-12}.$
(17) $\log 5.47 \times 10^{23}.$
(18) $\log 7.58 \times 10^{4}.$
(19) $\log 98.3$
(20) $\log 3140000.$

68. Interpolation. If there are more than three significant figures in the given number, its mantissa is not printed in the table; but it can be found approximately by the principle of proportional parts: *when a number is changed by an amount which is very small in comparison with the number itself, the change in the logarithm of the number is nearly proportional to the change in the number itself.*

For example, to find the logarithm of 37.68, we find from the table,

Mantissa of 3760 = 5752,
Mantissa of 3770 = 5763.

The difference between these mantissas, called the tabular difference, is 11. We note that an increase of 10 in 3760 produces an increase of 11 in its mantissa and we conclude that an increase of 8 in 3760 (to bring it up to 3768, the given digits) would produce an increase of $.8 \times 11 = 8.8$ in the mantissa. This number 8.8, called the correction, is to be added to the

mantissa of 3760, but in using a four place table we retain only four places in corrected mantissas, so here we add 9 (the integer nearest to 8.8); thus,

$$\begin{aligned} \log 37.60 &= 1.5752 \\ \text{correction} &= \quad\ 9 \\ \hline \log 37.68 &= 1.5761 \end{aligned}$$

Near the beginning of Table I. the tabular differences are so large as to make this process of interpolation inconvenient and in some instances unreliable. On this account there are printed on the third and fourth pages of Table I., *the mantissas of all four figure numbers whose first digit is* 1. By using these we can avoid interpolation at the beginning of the table. Thus, on the third page of the table we find,

$$\log 103.2 = 2.0137,$$

but if we find it by interpolation on the first page,

$$\log 103.2 = 2.0136$$

Example 1. Find the logarithm of .003467. Opposite 34 in column 6 find 5391; the tabular difference is 12; $.7 \times 12 = 8.4$; the mantissa is then $5391 + 8 = 5399$; hence $\log .003467 = 7.5399 - 10$.

Example 2. Find log 2.6582. Opposite 26 in column 5 find 4232; the tabular difference is 17; $.82 \times 17 = 13.9$; the mantissa is $4232 + 14 = 4246$; hence $\log 2.6582 = 0.4246$.

69. Accuracy of Results. The accuracy of results obtained by means of logarithms depends upon the number of decimal places given in the tables that are used, and this accuracy has reference to the significant figures counted from the left. In general, a table will give trustworthy results to as many significant figures, counted from the left, as there are decimal places given in the logarithms. For example, four-place logarithms would show no difference between 35492367 and 35490000.

Neither a four-place nor a five-place table would be of any use in financial computations where large sums are involved. It would take a nine-place table to yield exact results if the sums involved should reach a million dollars.

70. Reverse Reading of the Table. *To find the number when its logarithm is known.* This is sometimes called finding the *antilogarithm.* For this process we have the following rule.

Rule III. *If the mantissa is found exactly in the table, the first two figures of the corresponding number are found in the column N of the same row, while the third figure of the number is found at the top of the column in which the mantissa is found.*

Place the decimal point so that the rules in § 66 *are fulfilled.*

Example. Given $\log N = 1.7427$; to find N.

We find the mantissa 7427 in the row which has 55 in column N. The column in which 7427 is found has 3 at the top. Thus the significant figures in the number are 553. Since the characteristic is 1 we must have 2 figures to the left of the decimal point. Thus $N = 55.3$.

If the mantissa of the given logarithm is between two mantissas in the table, we may find the number whose logarithm is given by the following

Rule IV. *When the given mantissa is not found in the table, write down three digits of the number corresponding to the mantissa in the table next less than the given mantissa, determine a fourth digit by dividing the actual difference by the tabular difference, and locate the decimal point so that the rules for characteristics are fulfilled.*

Example. Given $\log N = 0.4675$; to find N.

The mantissa 4675 is not recorded in the table, but it lies between the two adjacent mantissas 4669 and 4683. The mantissa 4669 corresponds to the number 293. The tabular difference is 14. The actual difference between 4669 and 4675 is 6. The number 4675 is 6/14 of the interval from 4669 to 4683, and the corresponding number N is

about 6/14 of the way from 293 to 294, or, reducing 6/14 to a decimal, about .4 of a unit beyond 293. Hence the corresponding digits are 2934; hence $N = 2.934$.

The work may be written down as follows:

$$\log N = 0.4675$$
$$4669$$
$$14)60(4$$
$$N = 2.934$$

EXERCISES

Obtain the logarithm of each of the following numbers.

1. 3.1416	2. 1.732	3. 2.718
4. 1.414	5. 39.37	6. 0.4343
7. 3437	8. 0.0254	9. 0.9144
10. 0.003954	11. 0.016018	12. 0.0283
13. 7918.	14. 866500.	15. 92897000.

Find the antilogarithm of each of the following numbers.

16. 0.4563	17. 9 6378 − 10.	18. 5.3144
19. 1.7581	20. 8.2046 − 10.	21. 6.1126
22. 0.4971	23. 7.5971 − 10.	24. 4.9365
25. 4.6856 − 10.	26. 8.1530 − 10.	27. 8.6123 − 20.
28. 8.4048 − 10.	29. 8.4520 − 10.	30. 0.7318 − 20.

71. Cologarithms. *The* ***cologarithm*** *of a number is the logarithm of the reciprocal of the number.* (Compare (5) § 64.)

$$\text{Thus colog } 425 = \log \frac{1}{425} = \log 1 - \log 425$$

$$= 0 - 2.6284$$

But since we always wish to have the mantissa of a logarithm positive, we write $0 = 10 - 10$, and subtract 2.6284 from this, as follows:

$$\log 1 = 10.0000 - 10$$
$$\log 425 = \quad 2.6284$$
$$\text{colog } 425 = \quad 7.3716 - 10.$$

In practice this is done mentally by *beginning at the left not omitting the characteristic, and subtracting each digit from* 9, *except the last significant digit, which is subtracted from* 10.

In the process of division subtracting the logarithm of a number and adding its cologarithm are equivalent operations since dividing by N is equivalent to multiplying by its reciprocal.

72. Computation by Logarithms. It should be kept in mind that a logarithm is unchanged if at the same time any given number is added to and subtracted from it. This is useful in two cases:

First. When we wish to subtract a larger logarithm from a smaller;

Second. When we wish to divide a logarithm by an integer.

EXAMPLE 1. Find the value of $27.4 \div 652$.

$$\begin{aligned} \log 27.4 &= 1.4378 \\ &= 11.4378 - 10 \\ \log 652 &= 2.8142 \\ \hline \log x &= 8.6236 - 10 \\ x &= 0.04304 \end{aligned}$$

EXAMPLE 2. Find the value of $(0.0773)^{1/3}$.

$$\log 0.0773 = 8.8882 - 10.$$

It is convenient to have, after division by 3, -10 after the mantissa; hence, before dividing we add $20.0000 - 20$.

$$\begin{aligned} \log 0.0773 &= 28.8882 - 30 \text{ (divide by 3)}, \\ \log x &= 9.6294 - 10 \\ x &= 0.4250 \end{aligned}$$

EXAMPLE 3. Find the value of $\left[\dfrac{(42.6)(-3.14)}{62.4}\right]^{1/3}$.

We have no logarithms of negative numbers, but an inspection of this problem shows that the result will be negative and numerically

the same as though all the factors were positive; hence we proceed as follows:

$$\begin{aligned} \log 42.6 &= 1.6294 \\ \log 3.14 &= 0.4969 \\ \operatorname{colog} 62.4 &= 8.2048 - 10 \text{ (add)} \\ & \overline{\qquad\qquad} \\ & 3)0.3311 \qquad \text{(divide by 3)} \\ \log(-x) &= 0.1104 \\ -x &= 1.290, \text{ whence } x = -1.290. \end{aligned}$$

EXERCISES

Find approximate values of the following by aid of logarithms.

1. $231.6 \times .0036$.
2. $79 \times 470 \times 0.982$.
3. 13750×8799000.
4. $(-9503) \times (-0.008657)$.
5. $0.0356 \times (-0.00049)$.
6. 9.238×0.9152
7. $\dfrac{8075}{364.9}$.
8. $\dfrac{0.00542}{0.04708}$.
9. $\dfrac{24617}{-0.00054}$.
10. $\dfrac{67 \times 9 \times 0.462}{0.643 \times 7095}$
11. $\dfrac{9097 \times 5.408}{-225 \times 593 \times 0.8665}$.
12. $(2.388)^5$.
13. $(0.57)^{-4}$.
14. $(19/11)^8$.
15. $(1.014)^{25}$.
16. $\sqrt{67.54}$.
17. $\sqrt[3]{-0.3089}$.
18. $\sqrt[5]{(-9.718)^3}$.
19. $8^{5/4}$.
20. $(0.001)^{2/3}$.
21. $(29\frac{9}{25})^{3/2}$.
22. $(6\frac{2}{3})^{3.4}$.
23. $(-9306)^{3/7}$.
24. $(0.0067)^{2.5}$.
25. $\sqrt{\frac{5}{6}} \times \sqrt[3]{\frac{7}{45}}$.
26. $\dfrac{\sqrt{0.1}}{(0.009)^{3/5}}$.
27. $(0.00068)^{-5/4}$.
28. $\sqrt{\dfrac{854 \times \sqrt[3]{0.042}}{7.985 \times \sqrt[4]{0.0005}}}$.
29. $\sqrt[3]{\dfrac{7^{1/4} \times 92^{1/5} \times (0.01)^{1/2}}{(0.00026)^5 \times 5968^{1/3}}}$.
30. $\sqrt{\sqrt[6]{0.5804}\sqrt[3]{0.2405}}$.
31. $(6.89 \times 10^{-22})^{16/17}$.
Ans. 1.21×10^{-20}.
32. $(5.67 \times 10^{-18})^{9/11}$.
Ans. 7.76×10^{-5}.
33. $\dfrac{(1.3 \times 10^{28})(4.56 \times 10^{-21})}{\sqrt[734]{[(4.5 \times 10^{-7})10^{6.58}]^{100}}}$.
Ans. 5.51×10^{7}.
34. The amount a of a principal p at compound interest of rate r for n years is given by the formula: $a = p(1 + r)^n$. Find the amount of \$486 in 5 years at five per cent. ($r = .05$) if the interest is compounded annually. *Ans.* \$620.27

35. Find the amount of \$384 in 40 years at four per cent., interest compounded annually. *Ans.* \$1,843.59.

36. Find the simple interest on \$6,237.43 for 7 years at six per cent. Would the computation made with four-place logarithms, be sufficiently accurate for commercial purposes? Explain. *Ans.* \$2619.72.

37. The weight P in pounds which will crush a solid cylindrical cast-iron column is given by the formula

$$P = 98920 \frac{d^{3.55}}{l^{1.7}},$$

where d is the diameter in inches and l the length in feet. What weight will crush a cast-iron column 6 feet long and 4.3 inches in diameter? [Rietz and Crathorne] *Ans.* 834,200 lbs.

The area A in acres, of a triangular piece of ground, whose sides are a, b, c, rods, is given by the formula

$$A = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{160},$$

where $s = \frac{1}{2}(a + b + c)$. Compute the areas, in acres, of the following triangles:

38. $a = 127.6$, $b = 183.7$, $c = 201.3$.
39. $a = 0.9$, $b = 1.2$, $c = 1.5$.
40. $a = 408$, $b = 41$, $c = 401$.
41. $a = 63.89$, $b = 138.24$, $c = 121.15$.

42. The percentage earning power, E, of an individual, in so far as it depends upon the eyes is given by Magnus by the formula

$$E = C\sqrt{P_1}\sqrt[4]{M}\sqrt[x]{\frac{C_1 + C_2}{2}\sqrt{P_2}\sqrt[4]{M}},$$

where x takes one of the values 5, 7, or 10, C being the maximal central visual acuity, $\sqrt{P_1}$ the visual field, $\sqrt[4]{M}$ the action of the extrinsic muscles, C_1 and C_2 the central visual acuity of each eye, and $\sqrt{P_2}$ the peripheric vision. Compute the value of E if $C = 1$, $P_1 = 1$, $M = 1$, $C_1 = 1$, $C_2 = 0.58$, $x = 10$, $P_2 = 1$. *Ans.* 97.2%

43. Compute E if $C_1 = 0.41$, $C_2 = 0.25$, $x = 5$, $P_2 = M = P_1 = 1$, $C = 0.41$. *Ans.* 33.06%.

44. The percentage earning ability E, as dependent upon the eyes is given by Magnus as

$$E = FV\sqrt[x]{K},$$

where F = functional ability, V = necessary knowledge, K = the ability to compete (demand for him), x has one of the values 5, 7, or 10. Compute E for $F = 0.78792$, $V = 1$, $x = 10$, $K = 0.39396$.

Ans. 71.78%.

45. Compute E for $F = 0.8254$, $x = 10$, $K = 0.4127$, $V = 1$.

Ans. 75.52%.

46. When w grams of a substance is dissolved in v liters of water at $t°$ centigrade, the osmotic pressure, p, of the solution and the molecular weight, M, of the solute are connected by the equation

$$pv = 0.082\ (273 + t)w/M.$$

Compute the molecular weight of cane sugar from the data

(*a*) $p = 12.06$, $v = 1$, $t = 22.62$, $w = 171.0$ *Ans.* 343.7

(*b*) $p = 24.42$, $v = 3$, $t = 23.56$, $w = 102.6$ *Ans.* 340.5

Compute the osmotic pressure for glucose solution, given

(*c*) $v = 1$, $t = 26.90$, $w = 72$, $M = 180.21$ *Ans.* 9.824

(*d*) $v = 2$, $t = 22{,}20$, $w = 360$, $M = 178.46$ *Ans.* 24.36

73. The Slide Rule. The ***slide-rule*** is an instrument for carrying out mechanically the operations of multiplication and division. It is composed of two pieces, usually about the shape of an ordinary ruler; one of the pieces (called the ***slide***,) fits in a groove in the other piece. Each piece is marked in divisions

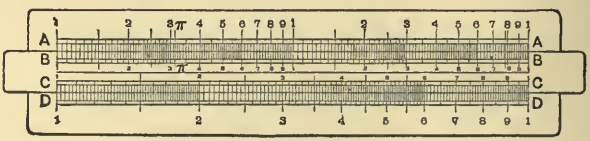


Fig. 22

(Fig. 22), such that the distance from one end (e. g., A) is equal to the logarithm of the number marked on it.

To multiply one number (e. g., 2.5) by another (e. g., 2) we

set the point marked 1 on scale B opposite the point marked 2.5 on scale A (see Fig. 23). Then the product appears on scale A

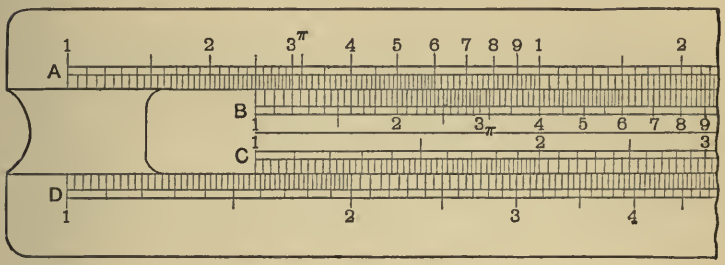


Fig. 23

opposite the point 2 on scale B. Thus 5 on scale A lies opposite 2 on scale B in Fig. 23. This follows from the fact that $\log 2.5 + \log 2 = \log 5$.

Likewise, if 1 on scale B is set opposite any number a on scale A, then we find opposite any number b on scale B the number ab on scale A.

Divisions can be performed by reversing this process. Thus if b on scale B be set opposite c on scale A, the 1 on scale B will be opposite c/b on scale A.

A little practice with such a slide-rule will make clear the actual procedure in any case.

Scales C and D are made just twice the size of scales A and B. It follows that any number on scale C, for example, is exactly opposite the square of that number on scale A. This facilitates the finding of squares and square roots, approximately.

Scales C and D may be used in place of scales A and B for multiplication and division. Indeed, after some practice, scales C and D will be preferred for this purpose.

More elaborate slide-rules, marked with several other scales, are for sale by all supply stores. Descriptions of these and full directions for their use will be found in special catalogs issued by instrument makers.

A simple slide-rule can be bought at a moderate price. One sufficient for temporary practice may be made by the student by cutting out the large figure printed on one of the fly-leaves of this book, and following the directions printed there.

The student should secure some form of slide-rule and he should use it principally in *checking* answers found by other processes.

As exercises the teacher may assign first very simple products and quotients. When the operation of the slide-rule has been mastered, the student may check the answers to the exercises on p. 86.

CHAPTER V

TRIGONOMETRY

74. Introduction. The sides and angles of a plane triangle are so related that any three given parts, provided at least one of them is a side, determine the shape and the size of the triangle.

Geometry shows how, from three such parts, to *construct* the triangle.

Trigonometry shows how to compute the unknown parts of a triangle from the numerical values of the given parts.

Geometry shows in a general way that the sides and angles of a triangle are mutually dependent. Trigonometry begins by showing the exact nature of this dependence in the *right triangle*, and for this purpose employs the *ratios of the sides.*

75. Definitions of Trigonometric Functions. Let A be any acute angle. Place it on a pair of axes as in Fig. 24, with the vertex at the origin, one side along the x-axis to the right, and the other side in the first quadrant. On this side choose any point M (except O) and drop MN perpendicular to the x-axis. Let $OM = r$; then by plane geometry,

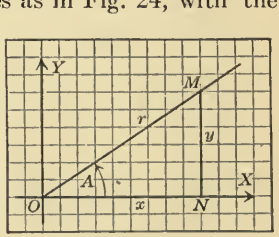


Fig. 24

$$r = \sqrt{x^2 + y^2},$$

where x and y are the coördinates of the point M. The different ratios of pairs of the three numbers x, y, and r, are designated as follows

(1) $\dfrac{y}{r} = \dfrac{\text{ordinate}}{\text{radius}}$ = the sine of angle A, written sin A,

(2) $\dfrac{x}{r} = \dfrac{\text{abscissa}}{\text{radius}}$ = the cosine of angle A, written cos A,

(3) $\dfrac{y}{x} = \dfrac{\text{ordinate}}{\text{abscissa}}$ = the tangent of angle A, written tan A.

The reciprocals of these rations are also used,

(4) $\frac{x}{y} = \frac{\text{abscissa}}{\text{ordinate}}$ = the cotangent of angle A, written ctn A,

(5) $\frac{r}{x} = \frac{\text{radius}}{\text{abscissa}}$ = secant of angle A, written sec A,

(6) $\frac{r}{y} = \frac{\text{radius}}{\text{ordinate}}$ = cosecant of angle A, written csc A.

These six ratios are called the ***trigonometric functions*** of the angle A. They do not at all depend upon the choice of the point M on the side of the angle but only upon the magnitude of the angle itself.

For if we choose any two points M' and M'' on the side of the

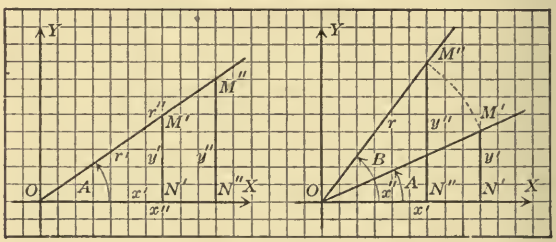


FIG. 25

same angle A, and denote their coördinates by (x', y') and (x'', y'') respectively, then by similar triangles,

$$\frac{y'}{r'} = \frac{y''}{r''} = \sin A, \qquad \frac{y'}{x'} = \frac{y''}{x''} = \tan A, \text{ etc.}$$

But if we take two points M' and M'' at the same distance r from O on the sides of two different angles A and B, then

$$\sin A = \frac{y'}{r} \neq \frac{y''}{r} = \sin B,$$

$$\tan A = \frac{y'}{x'} \neq \frac{y''}{x''} = \tan B,$$

and similarly the other functions of A and B are unequal.

From these definitions we deduce the following relations which are of fundamental importance in computing the unknown parts of right triangles.

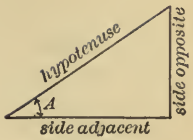


FIG. 26

In any right triangle, having fixed attention on one of the acute angles,

(7) The side opposite = hypotenuse × sine.
also = side adjacent × tangent.

(8) The side adjacent = hypotenuse × cosine.
also = side opposite × cotangent.

(9) $$\text{The hypotenuse} = \frac{\text{side opposite}}{\text{sine}}.$$
$$\text{also} = \frac{\text{side adjacent}}{\text{cosine}}.$$

EXERCISES

Find the six functions of each of the acute angles in the right triangle whose sides are:

1. 3, 4, 5.
2. 9, 40, 41.
3. 60, 91, 109.
4. 7, 24, 25.
5. 16, 63, 65.
6. 20, 99, 101.
7. 20, 21, 29.
8. 36, 77, 85.
9. 12, 35, 37.
10. $2n + 1$, $2n(n + 1)$, $2n^2 + 2n + 1$.
11. $2n$, $n^2 - 1$, $n^2 + 1$.
12. $2(n + 1)$, $n(n + 2)$, $n^2 + 2n + 2$.
13. $a(b^2 - c^2)$, $2abc$, $a(b^2 + c^2)$.

76. Functions of Complementary Angles. Let A and B be the acute angles in any right triangle. Then,

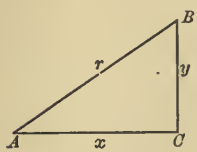


FIG. 27

$$\sin A = \cos B = \frac{y}{r}, \quad \cos A = \sin B = \frac{x}{r},$$
$$\tan A = \operatorname{ctn} B = \frac{y}{x}, \quad \operatorname{ctn} A = \tan B = \frac{x}{y},$$
$$\sec A = \csc B = \frac{r}{x}, \quad \csc A = \sec B = \frac{r}{y}.$$

Since $A + C = 90°$ (i. e., A and C are complementary), the above results may be stated in compact form as follows:

A function of an acute angle is equal to the co-function of its complementary acute angle.

77. Functions of 30°, 45°, 60°. On the sides of a right angle lay off unit distances AB and AC and draw BC, forming an isosceles right triangle, Fig. 28. The angles at B and C are each 45°, and the hypotenuse BC is equal to $\sqrt{2}$ (why?).

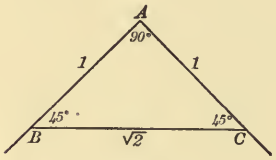


Fig. 28

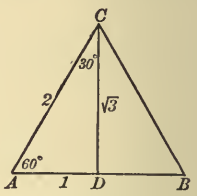


Fig. 29

From the definitions,

$$\sin 45° = \cos 45° = 1/\sqrt{2} = \sqrt{2}/2.$$
$$\tan 45° = \operatorname{ctn} 45° = 1.$$
$$\sec 45° = \csc 45° = \sqrt{2}/1 = \sqrt{2}.$$

Construct an equilateral triangle whose sides are 2 units long, Fig. 29. Bisect one of its angles forming a right triangle ACD, in which $A = 60°$, $C = 30°$, and the altitude CD is equal to $\sqrt{3}$ (why?). Then from the definitions,

$$\sin 60° = \cos 30° = \sqrt{3}/2. \qquad \operatorname{ctn} 60° = \tan 30° = 1/\sqrt{3}.$$
$$\cos 60° = \sin 30° = 1/2. \qquad \sec 60° = \csc 30° = 2.$$
$$\tan 60° = \operatorname{ctn} 30° = \sqrt{3}. \qquad \csc 60° = \sec 30° = 2/\sqrt{3}.$$

78. Eight Fundamental Relations. The following relations hold for the trigonometric functions of any acute angle A,

(10) $\sin A \csc A = 1$, sine and cosecant are reciprocals;

(11) $\cos A \sec A = 1$, cosine and secant are reciprocals;

(12) $\tan A \operatorname{ctn} A = 1$, tangent and cotangent are reciprorocals;

(13) $\tan A = \dfrac{\sin A}{\cos A}$; (14) $\operatorname{ctn} A = \dfrac{\cos A}{\sin A}$;

(15) $\sin^2 A + \cos^2 A = 1$;

(16) $\tan^2 A + 1 = \sec^2 A$; (17) $\operatorname{ctn}^2 A + 1 = \csc^2 A$.

These eight identities are fundamental relations and should be thoroughly learned by the student.

They may be proved as follows: (10), (11), (12) are direct consequences of the definitions in § 75. To prove (13), we have

$$\tan A = \frac{y}{x}, \quad \sin A = \frac{y}{r}, \quad \cos A = \frac{x}{r},$$

whence

$$\frac{\sin A}{\cos A} = \frac{y}{r} \div \frac{x}{r} = \frac{y}{x} = \tan A.$$

Similarly,

$$\frac{\cos A}{\sin A} = \frac{x}{r} \div \frac{y}{r} = \frac{x}{y} = \operatorname{ctn} A.$$

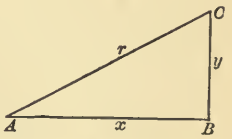


FIG. 30

From Fig. 30,

(18) $x^2 + y^2 = r^2$.

Dividing through by r^2, we have

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1,$$

whence $\cos^2 A + \sin^2 A = 1$.

Similarly, dividing (18) through by x^2, and then by y^2 we prove (16) and (17).

If the value of one function of an angle is known, the values of all the others can be found by means of these relations.

EXAMPLE. Given $\sin A = 1/2$. Then,

$$\cos A = \sqrt{1 - \sin^2 A} = \sqrt{\tfrac{3}{4}} = \tfrac{1}{2}\sqrt{3},$$

and, by (13),

$$\tan A = 1/\sqrt{3} = \tfrac{1}{3}\sqrt{3}.$$

Since the other three functions are reciprocals of these three, we have

$$\operatorname{ctn} A = \sqrt{3}, \qquad \sec A = \tfrac{2}{3}\sqrt{3}, \qquad \csc A = 2.$$

The values of these functions can also be found graphically by constructing a right triangle the ratio of whose sides are such as to make the sine of one angle equal to 1/2. This can evidently be done by making the side opposite equal to 1 and the hypotenuse equal to 2; then the side adjacent is equal to $\sqrt{3}$. (Why?) The other functions can now be read directly from the figure, using the definitions. Thus, tan A = side opposite ÷ side adjacent = $1/\sqrt{3} = \tfrac{1}{3}\sqrt{3}$.

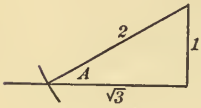


FIG. 31

EXERCISES

1. Given $\sin 40° = \cos 50°$; express the other functions of 40° in terms of functions of 50°.

2. The angles $45° + A$ and $45° - A$ are complementary; express the functions of $45° + A$ in terms of the functions of $45° - A$.

3. A and $90° - A$ are complementary; express the functions of $90° - A$ in terms of the functions of A.

4. Construct a right triangle, having given

(*a*) hypotenuse = 6, tangent of one angle = 3/2.
(*b*) cosine of one angle = 1/2, side opposite = 3.5.
(*c*) sine of one angle = 0.6, side adjacent = 2.
(*d*) cosecant of one angle = 4, side adjacent = 4.
(*e*) one angle = 45°, side adjacent = 20.
(*f*) one angle = 30°, side opposite = 25.

5. In Ex. 4, compute the remaining parts of each triangle.

6. Express each of the following as a function of the complementary angle:

(*a*) sin 30°. (*b*) tan 89°. (*c*) csc 18° 10′. (*d*) ctn 82° 19′.
(*e*) cos 45°. (*f*) ctn 15°. (*g*) cos 37° 24′. (*h*) csc 54° 46′.

7. Express each of the following as a function of an angle less than 45°:

(*a*) $\sin 60°$. (*b*) $\tan 57°$. (*c*) $\csc 69° 2'$. (*d*) $\operatorname{ctn} 89° 59'$.
(*e*) $\cos 75°$. (*f*) $\operatorname{ctn} 84°$. (*g*) $\cos 85° 39'$. (*h*) $\csc 45° 13'$.

8. Prove that if A is any acute angle,

(*a*) $\sin A \cdot \sec A = \tan A$. (*b*) $\sin A \cdot \operatorname{ctn} A = \cos A$.
(*c*) $\cos A \cdot \csc A = \operatorname{ctn} A$. (*d*) $\tan A \cdot \cos A = \sin A$.
(*e*) $\sin A \cdot \sec A \cdot \operatorname{ctn} A = 1$. (*f*) $\cos A \cdot \csc A \cdot \tan A = 1$.
(*g*) $(\sin A + \cos A)^2 = 1 + 2 \sin A \cos A$.
(*h*) $(\sec A + \tan A)(\sec A - \tan A) = 1$.
(*i*) $(1 + \tan^2 A) \sin^2 A = \tan^2 A$.
(*j*) $(1 - \sin^2 A) \csc^2 A = \operatorname{ctn}^2 A$.
(*k*) $\sin^4 A - \cos^4 A = \sin^2 A - \cos^2 A$.
(*l*) $\tan^2 A \cos^2 A + \cos^2 A = 1$.
(*m*) $(\sin A + \cos A)^2 + (\sin A - \cos A)^2 = 2$.
(*n*) $\sec A - \cos A = \sin A \tan A$.
(*o*) $(\sin^2 A - \cos^2 A)^2 = 1 - 4 \sin^2 A \cos^2 A$.
(*p*) $(1 - \tan^2 A)^2 = \sec^4 A - 4 \tan^2 A$.

9. Express the values of all the other functions of A in terms of (*a*) $\sin A$, (*b*) $\cos A$, (*c*) $\tan A$, (*d*) $\operatorname{ctn} A$, (*e*) $\sec A$, (*f*) $\csc A$.

79. Solution of Right Triangles. The values of the six trigonometric ratios have been computed for all acute angles, and recorded in convenient tables. They are given to four decimal places in Table II, at the end of the book. These tables, together with the definitions of the functions, enable us to solve all cases of right triangles.

80. General Directions for Solving Right Triangles.

(1) Draw a diagram approximately to scale, indicating the given parts. Mark the unknown parts by suitable letters, and estimate their values.

(2) *If one of the given parts is an acute angle,* consider the relation of the known parts to the one which it is desired to find, and apply the appropriate one of formulas (7), (8), (9), p. 93.

(3) *If two sides are given,* and one of the angles is desired,

think of the definition of that function of the angle which employs the two given sides.

(4) Check the results. The larger side must be opposite the larger angle, and the square of the hypotenuse must be equal to the sum of the squares of the other two sides.

The following examples illustrate the process of solution.

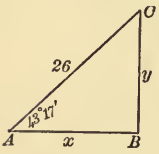


Fig. 32

Example 1. Given the hypotenuse = 26, and one angle = 43° 17′; find the two sides and the other acute angle. Do not use logarithms.

Draw a figure ABC in which $AC = 26$, $A = 43°\ 17'$ and denote the unknown parts by suitable letters, x, y, and C. Find C as the complement of A:

$$\begin{array}{r} 90°\ 00' \\ A = 43°\ 17' \\ \hline C = 56°\ 43' \end{array}$$

To find x note that it is adjacent to the given angle and that the hypotenuse is given,

Then by (8) § 75

$$x = 26 \cos 43°\ 17'$$
$$\cos 43°\ 17' = 0.7280$$
$$26$$
$$4368$$
$$1456$$
$$x = 18.928$$

Similarly by (7) § 75

$$y = 26 \sin 43°\ 17'$$
$$\sin 43°\ 17' = 0.6856$$
$$26$$
$$41136$$
$$13712$$
$$y = 17.8256$$

Check: $\tan A = y/x = 0.9418$. $\tan 43°\ 17' = 0.9418$.

Example 2. An acute angle of a right triangle is 62° 10′ and the opposite side is 78. Find the other parts. Solve by means of logarithms.

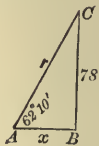


Fig. 33

By (8) § 75

$$x = 78 \operatorname{etn} 62°\ 10'$$
$$\log 78 = 11.8921 - 10$$
$$\log \operatorname{etn} 62°\ 10' = 9.7226 - 10$$
$$\log x = 1.6147$$
$$x = 41.18$$

By (9) § 75

$$r = 78/\sin 62°\ 10'$$
$$\log 78 = 11.8921 - 10$$
$$\log \sin 62°\ 10' = 9.9466 - 10$$
$$\log r = 1.9455$$
$$r = 88.20$$

CHECK:

$$\begin{aligned} r &= 88.20 \\ x &= 41.18 \\ \hline r + x &= 129.38 \\ r - x &= 47.02 \end{aligned}$$

$$\begin{aligned} \log (r + x) &= 2.1119 \\ \log (r - x) &= 1.6723 \\ \hline & 3.7842 \\ \log 78^2 &= 3.7842 \end{aligned}$$

EXAMPLE 3. The hypotenuse of a right triangle is 42.7 and one side is 18.5. Find the other parts. To find one of the angles, as C, note that the hypotenuse and side adjacent are known. Then

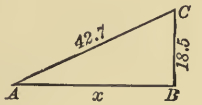


FIG. 34

$$\cos C = \frac{18.5}{42.7} = 0.4332$$

$$\begin{aligned} C &= 64^\circ\ 19'.6 \\ \overline{42.7}^2 &= 1823.29 \\ \overline{18.5}^2 &= 342.25 \\ \hline x^2 &= 1481.04 \\ x &= 38.48 \end{aligned}$$

$$A = 90^\circ - C = 25^\circ\ 40'.4$$

CHECK: $x = 18.5 \operatorname{ctn} 25^\circ\ 40'.4$

$$= 18.5 \times 2.0803 = 38.48$$

SOLUTION BY LOGARITHMS.

$$\cos C = \frac{18.5}{42.7}.$$

$$\begin{aligned} \log 18.5 &= 1.2672 \\ \log 42.7 &= 1.6304 - 10 \\ \hline \log \cos C &= 9.6368 \\ C &= 64^\circ\ 19' \end{aligned}$$

$$\begin{aligned} x^2 &= \overline{42.7}^2 - \overline{18.5}^2 \\ &= 61.2 \times 24.2 \\ \log 24.2 &= 1.3838 \\ \log 61.2 &= 1.7868 \\ \hline \log x^2 &= 3.1706 \\ \log x &= 1.5853 \\ x &= 38.48 \end{aligned}$$

81. Graphical Solution. As shown in § 35, if the triangle be drawn to scale, the unknown sides can be read off on the scale, and the unknown angles on a protractor. The results so obtained will be accurate enough to detect any large errors in the computations.

EXERCISES

Let A, B, C represent the three angles of any triangle and a, b, c the sides opposite these angles.

1. Solve graphically the following triangles:

(a) $a = 5$, $b = 4$, $c = 7$. *Ans.* $A = 44° 30'$, $B = 34°$, $C = 101° 30'$.

(b) $a = 3$, $b = 7$, $c = 8$. *Ans.* $A = 22°$, $B = 60°$, $C = 98°$.

(c) $a = 5$, $b = 7$, $c = 8$. *Ans.* $A = 38°$, $B = 60°$, $C = 82°$.

(d) $a = 8$, $b = 7$, $B = 60°$.

Ans. $A_1 = 82°$, $A_2 = 98°$, $C_1 = 38°$, $C_2 = 22°$, $c_1 = 5$, $c_2 = 3$.

(e) $a = 3$, $b = 5$, $c = 7$. *Ans.* $A = 22°$, $B = 38°$, $C = 120°$.

(f) $a = 7$, $A = 120°$, $b = 5$. *Ans.* $B = 38°$, $C = 22°$, $c = 3$.

(g) $a = 42$, $b = 51$, $A = 55°$.

Ans. $B_1 = 84°$, $B_2 = 96°$, $C_1 = 41°$, $C_2 = 29°$, $c_1 = 34$, $c_2 = 25$.

2. Solve the following right triangles ($C = 90°$).

	Given parts.	*Required parts (Answers).*		
(a)	$A = 30°$, $a = 12$,	$B = 60°$,	$b = 20.78$,	$c = 24$
(b)	$A = 45°$, $b = 8$,	$B = 45°$,	$a = 8$,	$c = 11.31$
(c)	$A = 60°$, $c = 20$,	$B = 30°$,	$a = 17.32$,	$b = 10$
(d)	$B = 25°$, $a = 72$,	$A = 65°$,	$b = 33.57$,	$c = 79.44$
(e)	$B = 40°$, $b = 33$,	$A = 50°$,	$a = 39.33$,	$c = 51.34$
(f)	$B = 70°$, $c = 81$,	$A = 20°$,	$a = 27.70$,	$b = 76.12$
(g)	$a = 6$, $b = 6$,	$A = 45°$,	$B = 45°$,	$c = 8.484$
(h)	$a = 3$, $c = 5$,	$A = 36° 52'$,	$B = 53° 8'$,	$b = 4$
(i)	$b = 12$, $c = 13$,	$A = 4° 46'$,	$B = 85° 14'$,	$a = 1$
(j)	$A = 23°$, $a = 3.246$,	$B = 67°$	$b = 7.647$,	$c = 8.307$
(k)	$A = 37°$, $b = 7.28$,	$B = 53°$,	$a = 5.486$,	$c = 9.116$
(l)	$B = 42°$, $c = 1021$,	$A = 48°$,	$a = 758.7$,	$b = 713.8$

3. The width of the gable of a building is 32 ft. 9 in. The height of the ridge of the roof above the plates is 14 ft. 6 in. Find the inclination of the roof, and the length of the rafters.

Ans. 41° 32′, 21 ft. 10 in.

4. The steps of a stairway have a tread of 10 in. and a rise of 7 in.; at what angle is the stairway inclined to the floor? *Ans.* 35°.

5. The shadow of a tower 200 ft. high is 252.5 ft. long. What is the angle of elevation of the sun?

6. A cord is stretched around two wheels with radii of 7 feet and 1 foot respectively, and with their centers 12 feet apart. Find the length of the cord. *Ans.* $12\sqrt{3} + 10\pi = 52.2$ ft.

7. Two objects A, B in a rectangular field are separated by a thicket. To determine the distance between them, the lines $AC = 45$ rods, $BC = 36$ rods, are measured parallel to the sides of the field. Find the distance AB. *Ans.* 57.63

8. One bank of a river is a bluff rising 75 ft. vertically above the water. The angle of depression of the water's edge on the opposite bank is 20° 27′. Find the width of the river. *Ans.* 201.1

9. A smokestack is secured by wires running from points on the ground 35 ft. from its base to points 3 ft. from its top. These wires are inclined at an angle of 40° to the ground. (*a*) What is the height of the smokestack? (*b*) The length of the wires? (*c*) What is the least number of wires necessary to secure the stack? If they are symmetrically placed, how far apart are their ground ends? (*d*) How far are the lines joining their ground ends from the foot of the stack? (*e*) From the top of the stack? (*f*) What angle do the wires make with these lines? (*g*) With each other? (*h*) What angle does the plane of two wires make with the ground? (*i*) What angle does the perpendicular from the foot of the stack on this plane make with the ground? (*j*) What is its length? [DURFEE]

10. A tree stands on a horizontal plane. At one point in this plane the angle of elevation of the top of the tree is 30°, at another point 100 feet nearer the base of the tree the angle of elevation of the top is 45°. Find the height of the tree.

11. Find the length of a ladder required to reach the top of a building 50 ft. high from a point 20 ft. in front of the building. What angle would the ladder in this position make with the ground?

82. General Angles. Rotation. Up to this point we have defined and used the trigonometric functions of acute angles only. Many problems require the consideration of obtuse angles and others, particularly those concerned with the rotating parts of machinery, involve angles greater than 180° or 360° even, and it is necessary to distinguish between parts in the same or parallel planes which rotate in the same or in opposite directions.

An angle may be thought of as being generated by the rotation of one of its sides about the vertex; its first position is called the *initial* side, its final position the *terminal* side of the angle. An angle generated by rotation opposite to the motion of the hands of a clock (*counterclockwise*) is said to be *positive*; an angle generated by *clockwise* rotation is said to be *negative*. In drawings a curved arrow may be used to show the direction of rotation, the arrow head indicating the terminal side.

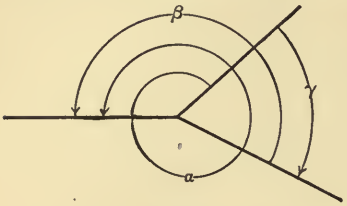


FIG. 35

83. Trigonometric Functions of any Angle. Let $\phi = XOP$ be any angle placed with its vertex at the origin and its initial side along the positive x-axis. Let P be any point (except O)

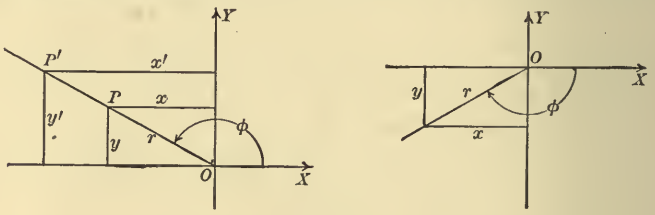


FIG. 36

on the terminal side and let x, y be its coördinates (positive, negative, or zero depending upon the position of P in the plane); let r be the distance from O to P (always positive). Then the trigonometric functions of ϕ are defined as follows:

$$\sin\phi = \frac{y}{r}, \qquad \cos\phi = \frac{x}{r}. \tag{19}$$

The definitions (19) apply to all angles without exception.

$$\tan\phi = \frac{y}{x}, \qquad \sec\phi = \frac{r}{x}. \tag{20}$$

The definitions (20) apply to all angles except odd multiples of a right angle; this exception is necessary because for all such angles x is zero.

$$\text{(21)} \qquad \operatorname{ctn}\phi = \frac{x}{y}, \quad \csc\phi = \frac{r}{y}.$$

The definitions (21) apply to all angles except even multiples of a right angle; for all such angles y is zero.

These definitions apply of course to all acute angles and give the same values as the definitions in § 75. These new definitions are more general because they apply to angles to which the former do not apply.

These ratios are independent of the choice of P on the terminal side of the given angle. They depend upon the magnitude and sign of the angle. For, if we choose a different point P' on the terminal side of ϕ, we shall have

$$\frac{x}{x'} = \frac{y}{y'} = \frac{r}{r'}$$

in magnitude and sign and this implies that

$$\frac{y'}{r'} = \frac{y}{r}, \qquad \frac{x'}{r'} = \frac{x}{r}, \qquad \frac{y'}{x'} = \frac{y}{x}, \qquad \text{etc.}$$

The signs of the trigonometric functions of an angle ϕ depend upon the *quadrant* of the plane in which the terminal side of ϕ falls when it is placed on the axes. An angle ϕ is said to be an angle in the first quadrant when its terminal side falls in that quadrant, and similarly for the second, third, and fourth quadrants. The signs of the sine and the cosine of an angle in each of the quadrants should be thoroughly learned. The accompanying diagram indicates these signs.

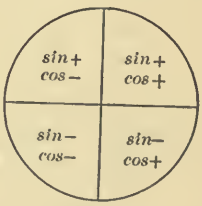


Fig. 37

The signs of the other functions are determined by noting that $\tan \phi$ is positive when $\sin \phi$ and $\cos \phi$ have *like* signs and negative when they have *unlike* signs; and that reciprocals have like signs.

84. The Fundamental Relations. The fundamental identities (10) to (18) which were proved for acute angles in § 78 are valid for any angle whatever. The proofs which are similar to those already given are left to the student.

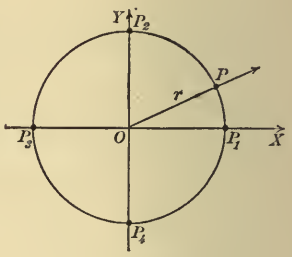


FIG. 38

85. Quadrantal Angles. Let P be a point on the terminal side of an angle ϕ at a distance r from the origin.

When $\phi = 0°$, P coincides with P_1 and its coördinates are $x = r$ and $y = 0$; then by § 83

$$\sin 0° = \frac{y}{r} = 0, \qquad \cos 0° = \frac{x}{r} = 1,$$

$$\tan 0° = \frac{y}{x} = 0, \qquad \sec 0° = \frac{r}{x} = 1.$$

The angle 0° has no cotangent nor cosecant.

When $\phi = 90°$, P coincides with P_2, $x = 0$, $y = r$; then

$$\sin 90° = \frac{y}{r} = 1, \qquad \cos 90° = \frac{x}{r} = 0,$$

$$\operatorname{ctn} 90° = \frac{x}{y} = 0, \qquad \csc 90° = \frac{r}{y} = 1.$$

The angle 90° has no tangent nor secant.

When $\phi = 180°$, P coincides with P_3, $x = -r$, $y = 0$; then

$$\sin 180° = \frac{y}{r} = 0, \qquad \cos 180° = \frac{x}{r} = -1,$$

$$\tan 180° = \frac{y}{x} = 0, \qquad \sec 180° = \frac{r}{x} = -1.$$

The angle 180° has no cotangent nor cosecant.

When $\phi = 270°$, P coincides with P_4, $x = 0$, $y = -r$; then

$$\sin 270° = \frac{y}{r} = -1, \qquad \cos 270° = \frac{x}{r} = 0,$$

$$\operatorname{ctn} 270° = \frac{x}{y} = 0, \qquad \csc 270° = \frac{r}{y} = -1.$$

The angle 270° has no tangent nor secant.

Often it is said that $\tan 90° = \infty$, but this does not mean that 90° has a tangent; it means that as an angle ϕ increases from 0° to 90°, $\tan \phi$ increases without limit, *and that before* ϕ *reaches* 90°. Similar remarks apply to the statements $\operatorname{ctn} 0° = \infty$, $\tan 270° = \infty$, etc.

86. Line Representations of the Trigonometric Functions. The trigonometric functions defined in § 83 are *abstract numbers*; each is the ratio of two lengths. They are not lengths nor lines. They can however very conveniently be represented by line segments in the sense that *the number of length units in the segment is equal to the magnitude of the function, and the sign of the segment is the same as the sign of the function.*

Let an angle ϕ of any magnitude and sign be placed on the axes, Fig. 39. With the origin as center and a radius one unit length draw a circle cutting the positive x-axis at A, the positive y-axis at B, and the terminal side of ϕ at P. Draw tangents to this circle at A and at B and produce the terminal side in one or both directions from O to cut these tangents in T and S respectively. Draw PQ perpendicular to the x-axis. Then, if we agree that QP shall be positive upward, OQ shall be positive to the right, and that OT, or OS, shall be positive when it has the same sense as OP and negative when it has the opposite sense,

QP represents $\sin \phi$, OQ represents $\cos \phi$,
AT represents $\tan \phi$, AS represents $\operatorname{ctn} \phi$,
OT represents $\sec \phi$, OS represents $\csc \phi$.

For, $\sin \phi = QP/OP$ = the number of units of length in QP since OP = unit length and $\sin \phi$ and QP agree in sign from quadrant to quadrant. Similarly the others may be proved.

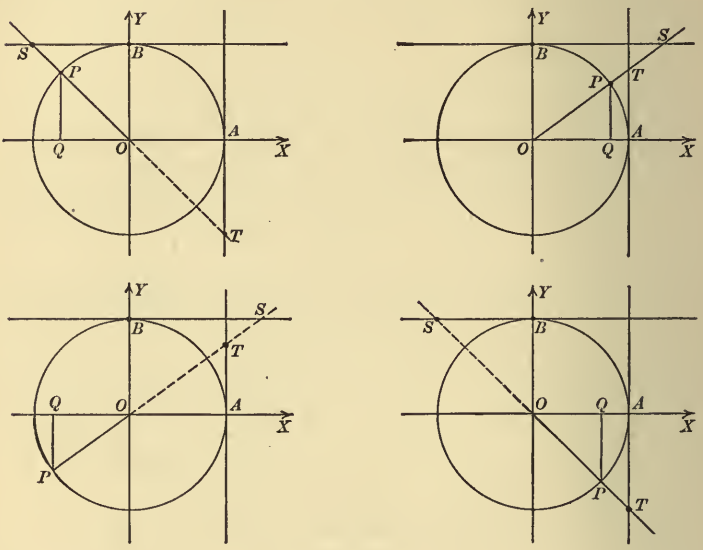


Fig. 39

The student is warned against thinking or saying that "QP is the sine of ϕ"; say "The number of units in QP is $\sin \phi$" or, "QP represents $\sin \phi$."

87. Congruent Angles. Any angle formed by adding to or subtracting from a given angle ϕ, any multiple of 360° is said to be congruent to ϕ; thus $-217°$ and 143° are congruent. It is obvious from the definitions and from the line representations of the functions of an angle that two congruent angles have equal functions. *The functions of any angle formed by adding to or subtracting from a given angle a multiple of* 360° *are the same as the corresponding functions of the given angle.*

88. Trigonometric Equations. To solve the equation $\sin x = 1/2$ is to find *all angles* which satisfy it. We know that $x = 30°$ is a solution for $\sin 30° = 1/2$; $x = 150°$, $x = -210°$, $x = 750°$, are also solutions. We can find all its solutions by the following graphical method.

1) To solve the equation

$$\sin x = s,$$

where s is a given number between -1 and $+1$, draw a unit circle center at the origin and on the y-axis lay off $OB = s$ (above O if $s > 0$, below if $s < 0$) and through B draw a parallel to the x-axis cutting the circle in C and D. Then the positive angles

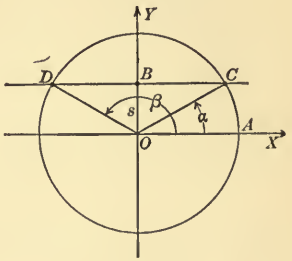


Fig. 40

$$\alpha = AOC \qquad \text{and} \qquad \beta = AOD$$

are solutions (and the only solutions between 0° and 360°) of the given equation. Any angle congruent to α or to β is also a solution, and there are no others. These results follow directly from the line representations of the functions in § 86.

2) To solve the equation

$$\cos x = c,$$

where c is a given number between -1 and $+1$, draw a unit circle center at the origin, Fig. 41, and lay off on the x-axis $OB = c$ (to the right if $c > 0$, to the left if $c < 0$) and draw through B a parallel to the y-axis cutting the circle in C and D. Then the positive angles

$$\alpha = AOC \qquad \text{and} \qquad \beta = AOD$$

are solutions (and the only solutions between 0° and 360°) of the given equation. Any angle congruent to α or to β is also a solution and there are no others.

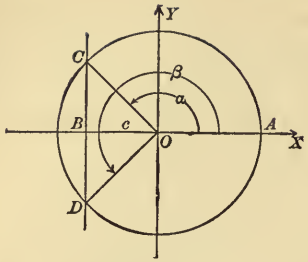


Fig. 41

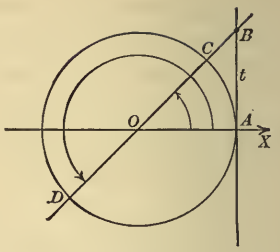


Fig. 42

3) To solve the equation

$$\tan x = t$$

where t is any given number whatever, draw a unit circle center at the origin, and lay off on the tangent at A,

$$AB = t$$

and draw a line through O and B cutting the circle in C and D. Then the positive angles

$$\alpha = AOC, \qquad \beta = AOD$$

are solutions (and the only solutions between 0° and 360°) of the given equation. Any angle congruent to α or to β is also a solution, and there are no others.

Many other trigonometric equations can be reduced to one of these three forms by the transformations given in § 78 and hence can be solved by the above methods.

For example, the equation

$$\operatorname{ctn} x = \tfrac{1}{3}$$

is equivalent to

$$\tan x = 3.$$

Again,

$$2 \sin^2 x - \cos x = 1$$

can be reduced to the form

$$(\cos x + 1)(\cos x - \tfrac{1}{2}) = 0$$

by replacing $\sin^2 x$ by $1 - \cos^2 x$, transposing all the terms to the left side, and factoring.

89. Graphs of the Trigonometric Functions. The variation in the sine of a given angle as the angle increases from 0° to 360° may be exhibited graphically as follows.

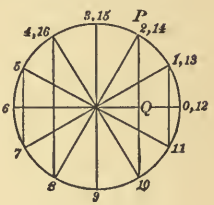


Fig. 43

Divide the circumference of a unit circle into a convenient number of equal arcs. In Fig. 43, the points of division are marked 0, 1, 2, 3, ··· 12. The length of the circumference is approximately 6.3; lay this off on the x-axis (Fig. 44) and divide it into the same number of equal parts and number them to correspond with the points of division on the circumference.

At each point of division on the x-axis lay off vertically the line representation QP, of the sine of the angle whose terminal side goes through the corresponding point of division on the circle. Connect the ends of these perpendiculars by a smooth curve. This is called the *sine curve* or the *graph of sin x*.

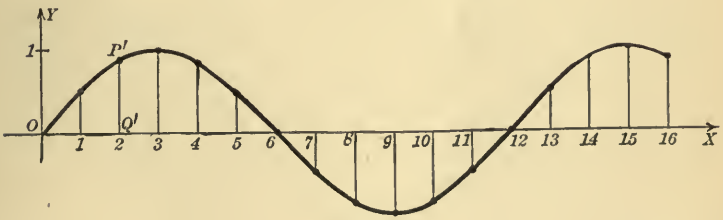


Fig. 44

As the angle increases from 0° to 360°, P moves along the circle successively through the points 0, 1, 2, 3, ···, 12, Q' moves along the x-axis successively through the corresponding points 0, 1, 2, 3, ···, 12, and P' traces the sine curve.

The graphs of the other trigonometric functions, cos x, tan x,

etc., are constructed in a similar manner by making use of their line representations given in § 86.

If the angle increases beyond 360°, P makes a second revolution around the circle, and the values of all the trigonometric functions repeat themselves in the same order and the graphs from $x = 6.3$ to $x = 12.6$ will in all cases be a repetition of those from $x = 0$ to $x = 6.3$. If P goes on indefinitely the graph will be repeated as many times as P makes revolutions.

Functions which repeat themselves as the variable or argument increases are called ***periodic functions.*** The ***period*** is the smallest amount of increase in the variable which produces the repetition of the value of the function. Thus, sin x is a periodic function with a period of 360°, while the period of tan x is 180°.

90. Functions of Negative Angles. Let $AOC = \phi$ be any angle placed on the axes; and let AOC' be its negative, $-\phi$;

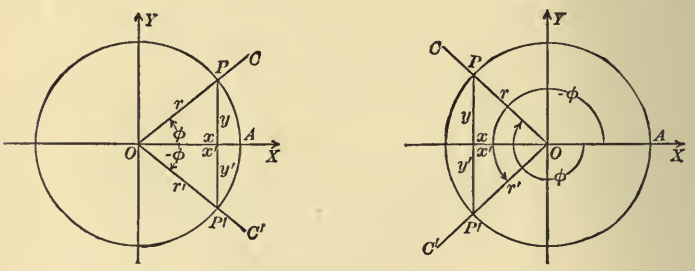


Fig. 45

lay off $OP' = OP$ and draw PP'. Let x, y be the coördinates of P and x', y' those of P'; let $OP = r$ and $OP' = r'$. Then *no matter what the magnitude or sign of* ϕ,

$$x = x', \qquad y = -y', \qquad r = r'$$

and by the definitions, § 83

$$\sin(-\phi) = \frac{y'}{r'} = -\frac{y}{r} = -\sin\phi,$$

$$\cos(-\phi) = \frac{x'}{r'} = \frac{x}{r} = \cos\phi,$$

$$\tan(-\phi) = \frac{y'}{x'} = -\frac{y}{x} = -\tan\phi,$$

$$\operatorname{ctn}(-\phi) = \frac{x'}{y'} = -\frac{x}{y} = -\operatorname{ctn}\phi,$$

$$\sec(-\phi) = \frac{r'}{x'} = \frac{r}{x} = \sec\phi,$$

$$\csc(-\phi) = \frac{r'}{y'} = -\frac{r}{y} = -\csc\phi.$$

91. The Trigonometric Functions of $90° + \phi$. Let any angle ϕ be placed on the axes; draw a circle, center at the origin, with any convenient radius r, cutting the terminal side of ϕ in P and the terminal side of $\phi + 90°$ in Q. Let the coördinates of P be (a, b); then no matter in what quadrant P is, Q is in the next quadrant and its coördinates are $(-b, a)$, for the right triangles OMP and QNO have the hypotenuse and an acute angle of the one equal to the hypotenuse and an acute angle of the other. Then by the definitions, § 83

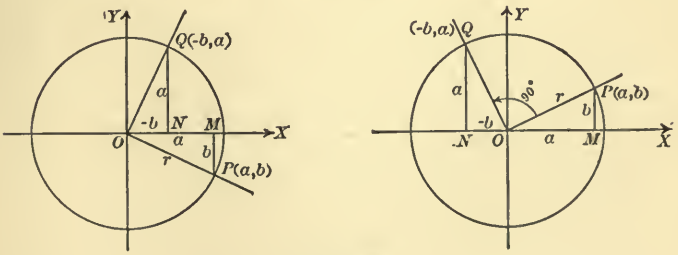


Fig. 46

$$\sin\ (90° + \phi) = \frac{a}{r} = \cos\phi$$

$$\cos\ (90° + \phi) = \frac{-b}{r} = -\sin\phi,$$

$$\tan\ (90° + \phi) = \frac{a}{-b} = -\operatorname{ctn}\phi,$$

$$\operatorname{ctn}\ (90° + \phi) = \frac{-b}{a} = -\tan\phi,$$

$$\sec\ (90° + \phi) = \frac{r}{-b} = -\csc\phi,$$

$$\csc\ (90° + \phi) = \frac{r}{a} = \sec\phi.$$

These formulas hold for all angles.*

92. Functions of $\pm\theta$, $90° \pm \theta$, $180° \pm \theta$, $270° \pm \theta$. If we put for ϕ in succession, $-\theta$, θ, $90° - \theta$, $90° + \theta$, $180° - \theta$, $180° + \theta$, $270° - \theta$, $270° + \theta$, we obtain the values in the following table, *θ being any angle.** By drawing diagrams the results tabulated can be verified. The student is advised to do this.

	$90°-\theta$.	$90°+\theta$.	$180°-\theta$.	$180°+\theta$.	$270°-\theta$.	$270°+\theta$.	$360°-\theta$.	$-\theta$.
sin	$\cos\theta$	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$	$-\sin\theta$	$-\sin\theta$
cos	$\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$	$-\sin\theta$	$\sin\theta$	$\cos\theta$	$\cos\theta$
tan	$\operatorname{ctn}\theta$	$-\operatorname{ctn}\theta$	$-\tan\theta$	$\tan\theta$	$\operatorname{ctn}\theta$	$-\operatorname{ctn}\theta$	$-\tan\theta$	$-\tan\theta$
ctn	$\tan\theta$	$-\tan\theta$	$-\operatorname{ctn}\theta$	$\operatorname{ctn}\theta$	$\tan\theta$	$-\tan\theta$	$-\operatorname{ctn}\theta$	$-\operatorname{ctn}\theta$
sec	$\csc\theta$	$-\csc\theta$	$-\sec\theta$	$-\sec\theta$	$-\csc\theta$	$\csc\theta$	$\sec\theta$	$\sec\theta$
csc	$\sec\theta$	$\sec\theta$	$\csc\theta$	$-\csc\theta$	$-\sec\theta$	$-\sec\theta$	$-\csc\theta$	$-\csc\theta$

If we inspect the table carefully, we find that it can be summed up in the two rules that follow.

*Except that no angle whose terminal side falls on the y-axis has a tangent or secant and no angle whose terminal side falls on the x-axis has a cotangent or cosecant.

1. *Determine the sign by the quadrant in which the angle would lie if θ were acute*; the result holds whether θ is acute or not.

2. *If* 90° *or* 270° *is involved, the function changes name to the corresponding cofunction, while if* 180° *or* 360° *is involved the function does not change name.*

EXAMPLE 1. $\sin 177° = \sin (180° - 3°) = +$ (rule 1) $\sin$ (rule 2) $3°$.
EXAMPLE 2. $\cos 177° = \cos (90° + 87°) = -$ (rule 1) $\sin$ (rule 2) $37°$.
EXAMPLE 3. $\tan 300° = \tan (180° + 120°) = +$ (rule 1) $\tan$ (rule 2) $120°$.

93. Plotting Graphs from Tables. For many purposes, such as the measurement of arcs and the speed of rotations, and generally in the calculus and higher mathematics, angles are measured in terms of a unit called the radian.

A ***radian*** is a positive angle such that when its vertex is placed at the center of a circle the intercepted arc is equal in length to the radius. This unit is thus a little less than one of the angles of an equilateral triangle, 57°.3 approximately. It is easy to change from radians to degrees and vice versa, by remembering that

(22) π **radians = 180 degrees.**

Unless some other unit is expressly stated, it is always understood that in graphs of the trigonometric functions the radian is the unit angle and that 1 unit on the x-axis represents 1 radian. These graphs can be constructed from a table of their values such as Table III at the end of the book. Thus to plot the graph of $\sin x$, draw a pair of rectangular axes on squared paper

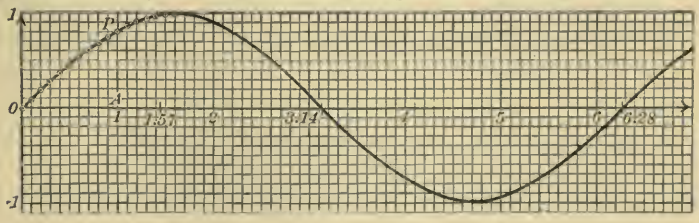


FIG. 47

and mark the points 1, 2, 3, $\cdots$ on the x-axis. These unit lengths are divided by the rulings of the cross-section paper into tenths. At each of these points of division on the x-axis lay off parallel to the y-axis the sine of the angle from the table, e. g., at 1 we plot $AP = .84 = \sin 1$ (radian). The curve may be extended beyond the first quadrant by the principles of § 92.

Similarly the graphs of $\cos x$ and $\tan x$ can be plotted from their tabulated values.

EXERCISES

1. Express each of the following functions as functions of angles less than 90°.

(a) sin 172°, (b) cos 100°, (c) tan 125°, (d) ctn 91°, (e) sec 110°,
(f) csc 260°, (g) sin 204°, (h) cos 359°, (i) tan 300°, (j) ctn 620°.

2. Express each of the preceding functions as functions of an angle less than 45°.

3. Express each of the following functions in terms of the functions of positive angles less than 45°.

(a) sin (− 160°), (b) cos (− 30°), (c) csc 92° 25′,
(d) sec 299° 45′, (e) sin (− 52° 37′), (f) cos (− 196° 54′),
(g) tan 269° 15′, (h) ctn 139° 17′, (i) sec (− 140°),
(j) ctn (− 240°), (k) csc (− 100°), (l) sin (− 300°),
(m) cos 117° 17′, (n) sin 143° 21′ 16″, (o) tan 317° 29′ 31″,
(p) ctn 90° 46′ 12″, (q) sec (− 135° 14′ 11″), (r) cos (− 428°).

4. Simplify each of the following expressions.

(a) $\sin (90° + x) \sin (180° + x) + \cos (90° + x) \cos (180° - x)$.
(b) $\cos (180° + x) \cos (270° - y) - \sin (180° + x) \sin (270° - y)$.
(c) $\sin 420° \cos 390° + \cos (-300°) \sin (-330°)$.

5. Prove each of the following relations.

(a) $\cos \frac{1}{3}(x - 270°) = +\sin x/3$.
(b) $\sec (-x - 540°) = -\sec x$.

6. Verify each of the following equations.

(a) $\cos 570° \sin 510° - \sin 330° \cos 390° = 0$.
(b) $\cos (90° + \alpha) \cos (270° - \alpha) - \sin (180° - \alpha) \sin (360° - \alpha) = 2 \sin^2 \alpha$.

(*c*) $3 \tan 210° + 2 \tan 120° = -\sqrt{3}$.

(*d*) $5 \sec^2 135° - 6 \operatorname{ctn}^2 300° = 8$.

(*e*) $\sin (90° + x) \sin (180° + x) + \cos (90° + x) \cos (180° - x) = 0$.

(*f*) $\dfrac{\sin (180° - \alpha)}{\sin (270° - \alpha)} \tan (90° + \alpha) + \csc^2 (270° - \alpha) = 1 + \sec^2 \alpha$.

7. Construct a table containing the functions of the eighths and twelfths of 360°.

8. In each of the following equations find graphically the two solutions which are between 0° and 360° and compute the values of the other five functions of each of these angles.

(*a*) $\sin x = 3/5$. (*b*) $\sin x = -1/3$. (*c*) $\cos x = -1/3$.
(*d*) $\operatorname{ctn} x = -3$. (*e*) $\sec x = -5/3$. (*f*) $\csc x = 13/5$.
(*g*) $\csc x = -\sqrt{3}$. (*h*) $\tan x = -\sqrt{7}$. (*i*) $\tan x = 2.5$.

9. Verify each of the following equations.

(*a*) $\sin 90° + \cos 180° = 0$.
(*b*) $\sin 270° + \cos 0° = 0$.
(*c*) $\csc 90° + \sec 180° = 0$.
(*d*) $\csc 270° + \sec 0° = 0$.
(*e*) $\sin 0° + \cos 270° = 0$.
(*f*) $\sin 180° + \cos 90° = 0$.
(*g*) $\sec 270° + \csc 0° = 0$.
(*h*) $\sin 120° + \sin 300° = 0$.
(*i*) $\cos 150° + \cos 330° = 0$.
(*j*) $\tan 135° + \tan 225° = 0$.
(*k*) $\operatorname{ctn} 315° + \operatorname{ctn} 45° = 0$.
(*l*) $\sin 120° + \cos 210° = 0$.

10. Find graphically another angle between 0° and 360° which has the same

(*a*) sine as 140°, (*b*) sine as 220°, (*c*) cosine as 330°,
(*d*) tangent as 230°, (*e*) cotangent as 110°, (*f*) secant as 160°.

11. Find the values of θ between 0° and 360° which satisfy the following equations.

(*a*) $\sin \theta = \sin 320°$.
(*b*) $\tan \theta = \tan 125°$.
(*c*) $\sec \theta = \sec 80°$.
(*d*) $\cos \theta = -\cos 50°$.
(*e*) $\operatorname{ctn} \theta = -\operatorname{ctn} 220°$.
(*f*) $\csc \theta = -\csc 340°$.

12. In what quadrant does an angle lie if sine and cosine are both negative? if cosine and tangent are both negative? if cotangent is positive and sine negative?

13. In finding $\cos x$ from the equation $\cos x = \pm \sqrt{1 - \sin^2 x}$, when must we choose the positive and when the negative sign?

14. Plot the graphs of each of the following functions and determine its period.

(*a*) $\cos x$. (*b*) $\tan x$. (*c*) $\operatorname{ctn} x$.
(*d*) $\sec x$. (*e*) $\csc x$. (*f*) $\sin(-x)$.
(*g*) $\cos(-x)$. (*h*) $\sin(90° + x)$. (*i*) $\sin x - \cos x$.

15. Plot the graph of each of the following functions.

(*a*) $x + \sin x$. (*b*) $x^2 + \sin x$. (*c*) $\sin x + \cos x$.
(*d*) $x + \cos x$. (*e*) $x - \cos x$. (*f*) $x - 1 + \sin x$.

94. Sine and Cosine of the Sum of two Angles. Let $AOB = x$, $BOC = y$, then $AOC = x + y$. With O as center and a convenient radius $r > 0$, strike an arc cutting OC in P. Drop PQ perpendicular to OB, also PR and QS perpendicular to

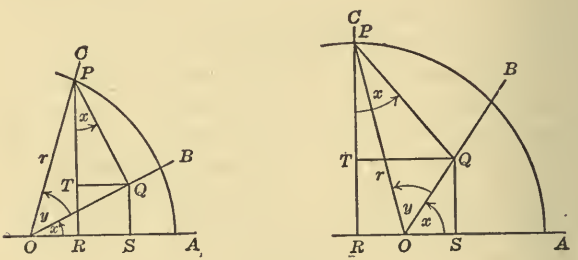


FIG. 48

OA. Through Q draw a parallel to OA cutting PR in T. Then by (7), § 75,

$$r \sin(x + y) = RP = SQ + TP.$$

Now by (7) and (8), § 75, we have

$$OQ = r \cos y \text{ and } SQ = OQ \sin x = r \cos y \sin x,$$
$$PQ = r \sin y \text{ and } TP = PQ \cos x = r \sin y \cos x.$$

Hence we may write

$$r \sin(x + y) = r \cos y \sin x + r \sin y \cos x,$$

and

(23) $$\sin(x + y) = \sin x \cos y + \cos x \sin y.$$

Similarly, we may write

$$r \cos (x + y) = OR = OS - TQ.$$

Then as before,

$$OS = OQ \cos x = r \cos y \cos x,$$
$$TQ = PQ \sin x = r \sin y \sin x.$$

Hence we may write

$$r \cos (x + y) = r \cos y \cos x - r \sin y \sin x,$$

and

$$\cos (x + y) = \cos x \cos y - \sin x \sin y. \tag{24}$$

The above formulas, therefore, hold true for all acute angles x and y. They are called the ***addition formulas.***

It is readily proved that if $x = \alpha$ and $y = \beta$ are any two acute angles for which these formulas hold good they will hold good for any two of the angles α, β, $\alpha + 90°$, $\alpha - 90°$, $\beta + 90°$, $\beta - 90°$. Therefore, since we have found that they hold good for all acute angles, they hold good for all positive or negative angles of any magnitude whatever.

The addition formulas may be translated into words as follows:

I. *The sine of the sum of two angles is equal to the sine of the first times the cosine of the second, plus the cosine of the first times the sine of the second.*

II. *The cosine of the sum of two angles is equal to the cosine of the first times the cosine of the second minus the sine of the first times the sine of the second.*

95. Tangent of the Sum of two Angles. This can be derived from the addition formulas as follows

$$\tan (x + y) = \frac{\sin (x + y)}{\cos (x + y)} = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}.$$

If we divide each term of the numerator and denominator of

the last fraction by $\cos x \cos y$, we have

$$\tan (x + y) = \frac{\dfrac{\sin x}{\cos x} + \dfrac{\sin y}{\cos y}}{1 - \dfrac{\sin x \sin y}{\cos x \cos y}},$$

that is

$$\tan (x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}. \tag{25}$$

This formula holds good for all angles such that x, y, and $x + y$ have tangents.

96. Functions of Twice an Angle. If we put x for y in (23), (24), § 94, and (25), § 95, these formulas give

$$\sin 2x = 2 \sin x \cos x. \tag{26}$$

$$\cos 2x = \cos^2 x - \sin^2 x. \tag{27}$$

$$= 2 \cos^2 x - 1. \tag{28}$$

$$= 1 - 2 \sin^2 x. \tag{29}$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}. \tag{30}$$

97. Functions of Half an Angle. The preceding formulas are true for all values of x for which they have a meaning. Hence we may replace x by any other quantity. If we write $x/2$ in place of x in (28) and (29), § 96, and solve the resulting equations for $\sin (x/2)$ and $\cos (x/2)$, we find

$$\sin \tfrac{1}{2}x = \pm \sqrt{\frac{1 - \cos x}{2}}. \tag{31}$$

$$\cos \tfrac{1}{2}x = \pm \sqrt{\frac{1 + \cos x}{2}}. \tag{32}$$

Whence on dividing (31) by (32)

$$\tan \tfrac{1}{2}x = \pm \sqrt{\frac{1 - \cos x}{1 + \cos x}} = \frac{1 - \cos x}{\sin x} = \frac{\sin x}{1 + \cos x}. \tag{33}$$

The positive or negative sign is to be chosen according to the quadrant in which $x/2$ lies.

EXERCISES

1. Putting $75° = 45° + 30°$, find $\cos 75°$ and $\tan 75°$.
2. Putting $15° = 45° + (-30°)$, find $\sin 15°$, $\cos 15°$, and $\tan 15°$.
3. Putting $15° = 60° + (-45°)$, find $\sin 15°$, $\cos 15°$, and $\tan 15°$.
4. Putting $90° = 60° + 30°$, find $\sin 90°$ and $\cos 90°$.
5. Show that $\sin(x - y) = \sin x \cos y - \cos x \sin y$.
6. Show that $\cos(x - y) = \cos x \cos y + \sin x \sin y$.
7. Putting $15° = 60° - 45°$, find $\sin 15°$.
8. Show that $\sin 3x = \sin x(3 - 4\sin^2 x) = \sin x(4\cos^2 x - 1)$.
9. Show that $\cos 3x = \cos x(4\cos^2 x - 3) = \cos x(1 - 4\sin^2 x)$.
10. Find $\sin 4x$; $\cos 4x$; $\tan 4x$.
11. Show that $\tan(45° + A) = \dfrac{1 + \tan A}{1 - \tan A}$.
12. Show that

(*a*) $$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y},$$

(*b*) $$\operatorname{ctn}(x + y) = \frac{\operatorname{ctn} x \operatorname{ctn} y - 1}{\operatorname{ctn} x + \operatorname{ctn} y}.$$

13. From the trigonometric ratios of 30°, find $\sin 60°$, $\cos 60°$, $\tan 60°$.
14. Express $\sin 6A$, $\cos 6A$, $\tan 6A$ in terms of functions of $3A$.
15. Find $\sin 22\frac{1}{2}°$, $\cos 22\frac{1}{2}°$, and $\tan 22\frac{1}{2}°$, from $\cos 45°$.
16. Find $\sin 15°$, $\cos 15°$, and $\tan 15°$, from $\cos 30°$.
17. Find $\cos(x + y)$, having given $\sin x = 3/5$ and $\sin y = 5/13$, x being positive acute, y being positive obtuse. *Ans.* $-63/65$.
18. Verify the following:

(*a*) $\sin(60° + x) - \sin(60° - x) = \sin x$.

(*b*) $\cos(30° + y) - \cos(30° - y) = -\sin y$.

(*c*) $\cos(45° + x) + \cos(45° - x) = \sqrt{2}\cos x$.

(*d*) $\cos(Q + 45°) + \sin(Q - 45°) = 0$.

(*e*) $\sin(x + y)\sin(x - y) = \sin^2 x - \sin^2 y$.

(*f*) $\dfrac{\sin(x + y)}{\sin(x - y)} = \dfrac{\tan x + \tan y}{\tan x - \tan y}$.

(*g*) $\sin 2x = \dfrac{2\tan x}{1 + \tan^2 x}$.

(*h*) $\sec 2x = \dfrac{\csc^2 x}{\csc^2 x - 2}$.

(*i*) $\tan \frac{1}{4}x = \dfrac{\sin \frac{1}{2}x}{1 + \cos \frac{1}{2}x}$.

(*j*) $\operatorname{ctn} \frac{1}{4}x = \dfrac{\sin \frac{1}{2}x}{1 - \cos \frac{1}{2}x}$.

(*k*) $\tan \frac{1}{2}A = \dfrac{1 - \cos A}{\sin A}$.

(*l*) $2 \csc 2s = \sec s \csc s$.

(*m*) $\tan (x + 45°) + \operatorname{ctn} (x - 45°) = 0$.

19. Prove each of the following identities.

(*a*) $\cos (A + B) \cos (A - B) = \cos^2 A - \sin^2 B$.

(*b*) $\sin (A + B) \cos B - \cos (A + B) \sin B = \sin A$.

(*c*) $\sin (A + B) + \cos (A - B) = (\sin A + \cos A)(\sin B + \cos B)$.

(*d*) $\cos^4 A = \frac{3}{8} + \frac{1}{2} \cos 2A + \frac{1}{8} \cos 4A$.

(*e*) $\sin^4 A = \frac{3}{8} - \frac{1}{2} \cos 2A + \frac{1}{8} \cos 4A$.

(*f*) $\sin^2 A \cos^2 A = \frac{1}{8} - \frac{1}{8} \cos 4A$.

(*g*) $\sin^2 A \cos^4 A = \frac{1}{16} + \frac{1}{32} \cos 2A - \frac{1}{16} \cos 4A - \frac{1}{32} \cos 6A$.

(*h*) $\cos (x - y + z) = \cos x \cos y \cos z + \cos x \sin y \sin z - \sin x \cos y \sin z + \sin x \sin y \cos z$.

(*i*) $\cos x \sin (y - z) + \cos y \sin (z - x) + \cos z \sin (x - y) = 0$.

(*j*) $\sin A + \sin B = 2 \sin \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B)$.

(*k*) $\sin A - \sin B = 2 \cos \frac{1}{2}(A + B) \sin \frac{1}{2}(A - B)$.

(*l*) $\cos A + \cos B = 2 \cos \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B)$.

(*m*) $\cos A - \cos B = -2 \sin \frac{1}{2}(A + B) \sin \frac{1}{2}(A - B)$.

(*n*) $\sin A \cos (B - C) - \sin B \cos (A - C) = \sin (A - B) \cos C$.

(*o*) $\cos^2 \frac{1}{2}\phi(1 + \tan \frac{1}{2}\phi)^2 = 1 + \sin \phi$.

(*p*) $\sin^2 \frac{1}{2}x(\operatorname{ctn} \frac{1}{2}x - 1)^2 = 1 - \sin x$.

(*q*) $\sec^2 \frac{1}{2}A = \dfrac{2 \sec A}{1 + \sec A}$. (*r*) $\csc^2 \frac{1}{2}A = \dfrac{2 \sec A}{\sec A - 1}$.

98. Solution of Oblique Triangles. One of the chief uses of trigonometry is to solve triangles. That is, having given three parts of a triangle (sides and angles) at least one of which must be a side, to find the others. In plane geometry it has been shown how to construct a triangle, having given

Case I. *Two angles and one side.*

Case II. *Two sides and the angle opposite one of them.*

Case III. *Two sides and the included angle.*

Case IV. *Three sides.*

When the required triangle has been constructed by scale and protractor the parts not given may be found by actual measurement. The results obtained by such graphic methods are not, however, sufficiently accurate for many practical purposes.

Nevertheless, they are very useful as a check upon the computed values of the unknown parts. Other checks are furnished by the theorems of plane geometry that the sum of the angles of any triangle is 180°, and that if two sides (angles) are unequal the greater side (angle) lies opposite the greater angle (side). The properties of isosceles triangles can also be used in certain special cases.

The direction *solve a triangle* tacitly assumes that a sufficient number of parts of an actual triangle are given. A proposed problem may violate this assumption and there will be no solution. Thus, there is no triangle whose sides are 14, 24, and 40; likewise, there is no triangle of which two sides are 9 and 10 and the angle opposite the former is 64° 10′. *Any triangle which can be constructed can be solved.*

Any oblique triangle can be divided into right triangles by a perpendicular from a vertex upon the opposite side, and this method when applied to the various cases leads to three laws, called the law of *sines*, the law of *cosines*, and the law of *tangents*, by means of which the unknown parts of any oblique triangle can be computed. We proceed to prove these laws.

99. Law of Sines. *Any two sides of a triangle are to each other as the sines of the opposite angles.*

In any oblique triangle let a, b and c be the measures of the lengths of the sides and A, B, and C the measures of the angles opposite. Drop the perpendicular $CD = p$ from the vertex of angle C to the opposite side.

Two possible cases are shown in Figs. 49, 50. In either of these figures,

$$p = b \sin A.$$

In Fig. 49,

$$p = a \sin B.$$

In Fig. 50,

$$p = a \sin (180° - B) = a \sin B.$$

Therefore, whether the angles are all acute, or one is obtuse

$$a \sin B = b \sin A,$$

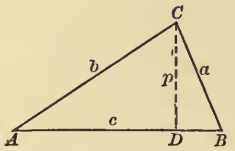


FIG. 49

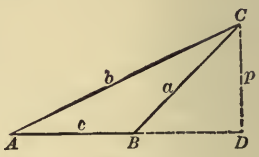


FIG. 50

whence dividing first by $\sin A \sin B$, and second by $b \sin B$,

$$\frac{a}{\sin A} = \frac{b}{\sin B}, \quad \text{or} \quad \frac{a}{b} = \frac{\sin A}{\sin B}. \tag{34}$$

Similarly, by drawing perpendiculars from A and B to the opposite sides, we obtain

$$\frac{b}{\sin B} = \frac{c}{\sin C}, \qquad \frac{a}{\sin A} = \frac{c}{\sin C}.$$

Hence,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}. \tag{35}$$

It is evident that a triangle may be solved by the aid of the law of sines if two of the three known parts are a side and its opposite angle. The case of two angles and the included side being given, may also be brought under this head, since we may find the third angle which lies opposite the given side.

100. Law of Cosines. *In any triangle, the square of any side is equal to the sum of the squares of the other two sides minus twice the product of these two sides into the cosine of their included angle.*

Let ABC be any triangle. Drop a perpendicular BD from B on AC or AC produced. Two possible cases are shown in

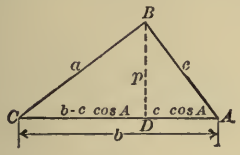


Fig. 51

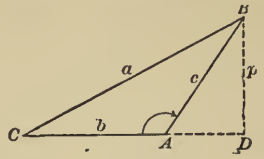


Fig. 52

Figs. 51, 52. Then we have either

$$CD = b - AD \text{ (Fig. 51)},$$
$$= b - c \cos A,$$

or else

$$CD = b + AD \text{ (Fig. 52)}$$
$$= b + c \cos (180° - A)$$
$$= b - c \cos A,$$

and

$$p = c \sin A \text{ (Fig. 51)},$$
$$p = c \sin (180° - A) = c \sin A \text{ (Fig. 52)}.$$

Hence, in either figure, we may write

$$CD = b - c \cos A \qquad \text{and} \qquad p = c \sin A.$$

Again, in either figure,

$$a^2 = \overline{CD}^2 + p^2$$
$$= (b - c \cos A)^2 + (c \sin A)^2$$
$$= b^2 - 2bc \cos A + c^2 (\sin^2 A + \cos^2 A)$$
$$= b^2 - 2bc \cos A + c^2,$$

that is

(36) $$a^2 = b^2 + c^2 - 2bc \cos A.$$

In like manner it may be proved that the law of cosines applies to the side b or to the side c.

These formulas may be used to find the angles of a triangle when the three sides are given and also to find the third side when two sides and the included angle are given.

101. Law of Tangents. *The sum of any two sides of a triangle is to their difference as the tangent of half the sum of their opposite angles is to the tangent of half their difference.*

From the law of sines, we have

$$\frac{a}{b} = \frac{\sin A}{\sin B};$$

whence, by division and composition in proportion, we find

$$\frac{a+b}{a-b} = \frac{\sin A + \sin B}{\sin A - \sin B}. \tag{37}$$

Let $x + y = A$ and $x - y = B$. Then we have

$$2x = A + B, \quad \text{and} \quad x = \tfrac{1}{2}(A + B),$$
$$2y = A - B, \quad \text{and} \quad y = \tfrac{1}{2}(A - B).$$

Hence, substituting in (37), we find

$$\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\sin (x+y) + \sin (x-y)}{\sin (x+y) - \sin (x-y)}$$
$$= \frac{2 \sin x \cos y}{2 \cos x \sin y} = \frac{\tan x}{\tan y}$$
$$= \frac{\tan \tfrac{1}{2}(A+B)}{\tan \tfrac{1}{2}(A-B)}.$$

From (37) and the preceding result, we have

$$\frac{a+b}{a-b} = \frac{\tan \tfrac{1}{2}(A+B)}{\tan \tfrac{1}{2}(A-B)}. \tag{38}$$

Since

$\tan \frac{1}{2}(A + B) = \tan \frac{1}{2}(180° - C) = \tan (90° - \frac{1}{2}C) = \operatorname{ctn} \frac{1}{2}C$,

we may write the law of tangents in the form

$$\tan \tfrac{1}{2}(A - B) = \frac{a-b}{a+b} \operatorname{ctn} \tfrac{1}{2}C. \tag{39}$$

As a check, (38) is the more convenient form, while for solving triangles, (39) is preferred by some computers. If $b > a$, then $B > A$. The formula is still true, but to avoid negative numbers the formula in this case should be written in the form

$$\frac{b + a}{b - a} = \frac{\tan \frac{1}{2}(B + A)}{\tan \frac{1}{2}(B - A)}. \tag{40}$$

When two sides and the included angle are given, as a, b, C, the law of tangents may be employed in finding the two unknown angles A and B.

102. Methods of Computation. The method to be used in computing the unknown parts of a triangle depends on what parts are given. In what follows triangles are classified according to the given parts and the methods of computation are stated and illustrated by examples.

103. Case I. Given two Angles and one Side. There is always one and only one solution, provided the sum of the given angles is less than 180°.

The third angle is found by subtracting the sum of the two given angles from 180°. The unknown sides are found, successively, by the law of sines.

EXAMPLE. In a triangle given two angles 38° and 75° 43′, and the side opposite the former 180; find the other parts.

Construct the triangle approximately to scale and denote the unknown parts by suitable letters as in Fig. 53.

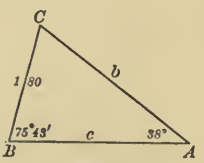


FIG. 53

First compute the third angle $C = 66° 17′$.

To compute b use the law of sines,

$$\frac{b}{180} = \frac{\sin 75° 43′}{\sin 38°}.$$

In any proportion *imagine* the means and the extremes to be paired by lines crossing at the equal sign,

$$\frac{2}{x} = \frac{3}{5}; \quad \frac{7}{11} = \frac{y}{4};$$

then the rule: *Multiply the pair of knowns and divide by the known in the other pair;* or, *Add the logarithms of the pair of knowns and the co-logarithm of the known in the other pair.*

First method: *without logarithms.*

$$\begin{array}{r} \sin 75^\circ\ 43' = \quad 0.9691 \\ \qquad\qquad\qquad\quad 180 \\ \hline \qquad\qquad\quad 775280 \\ \qquad\qquad\quad\ 9691 \\ \hline \sin 38^\circ = 0.6157)174.4380(283.3 \\ \qquad\qquad\quad\ 12314 \\ \hline \qquad\qquad\quad\ 51298, \text{ etc.} \end{array}$$

whence $b = 283.3$.

Second method: *with logarithms.*

$$\begin{array}{rl} \log 180 = & 2.2553 \\ \log \sin 75^\circ\ 43' = & 9.9864 - 10 \\ \operatorname{colog} \sin 38^\circ = & 0.2107 \\ \hline \log b = & 2.4524 \\ & \quad 18 \\ \hline & 15)60(4 \end{array} \qquad b = 283.4.$$

Similarly we may compute c. Using logarithms, we find $c = 267.7$. Not using logarithms, we find 267.6. The difference in the two answers is due to the slight inaccuracy caused by our using only four decimal places.

EXERCISES

1. Given two angles 43° and 67° and the included side 51; find the other parts. *Ans.* 70°, 49.96, 37.02.

2. Given two angles 24° 14′ and 43° 13′ and the side opposite the latter 240; find the other parts. *Ans.* 112° 33′, 143.9, 323.8.

3. Solve the triangle ABC being given $A = 17^\circ\ 17'$, $B = 102^\circ\ 25'$, and $a = 36.84$. *Ans.* $C = 60^\circ\ 18'$, $c = 107.7$, $b = 121.1$.

4. Solve the triangle LMN being given $L = 28^\circ$, $M = 51^\circ$, $l = 6.3$. *Ans.* $N = 101^\circ$, $n = 13.17$, $m = 10.43$.

104. Case II. Given two Sides and the Angle opposite one of Them. This case sometimes admits two solutions and on this account is called the ***ambiguous case.*** The number of solutions can be determined by constructing the triangle to scale as follows.

To fix our ideas, let the given angle be A, the given opposite side a, and the given adjacent side b. Construct the given angle A, and on one of its sides lay off $AC = b$, the given adjacent side, and drop a perpendicular CP, of length p, from C to the other side of the given angle A. With C as center and with radius a, the given opposite side, strike an arc to determine the vertex of the third angle B. Several possible cases are shown in Fig. 54.

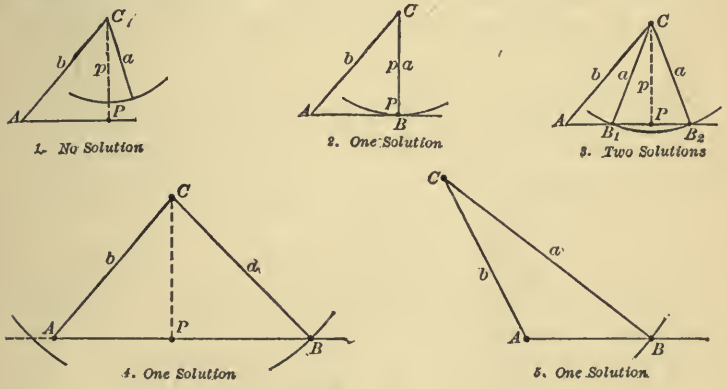


FIG. 54

A study of these diagrams shows that there will be two solutions when, and only when, the given angle is acute and the length of the given opposite side is intermediate between the lengths of the perpendicular and the given adjacent side; that is

$$A < 90° \qquad \text{and} \qquad p < a < b.$$

The two triangles to be solved are AB_1C and AB_2C. Since

the triangle B_1CB_2 is isosceles, the obtuse angle B_1 (i. e., angle AB_1C) is the supplement of the acute angle B_2.

The following examples illustrate the method of computing the unknown parts in Case II.

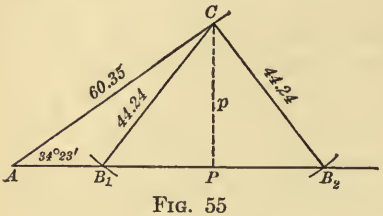


FIG. 55

EXAMPLE 1. One angle of a triangle is 34° 23′, the side opposite is 44.24 and another side is 60.35; find the other parts.

On constructing the triangle to scale as in Fig. 55, it appears that there are two solutions. This is verified by computing $p = 60.35 \sin 34° 23'$. Noting from the tables that $\sin 35° < .6$, it is evident that $p < 40$.

Let us solve first the triangle AB_2C, the angle B_2 being acute. By the law of sines,

$$\frac{60.35}{44.24} = \frac{\sin B_2}{\sin 34° 23'}$$

$$B_2 = 50° 23'$$

$$\begin{aligned} \log 60.35 &= 1.7807 \\ \log \sin 34° 23' &= 9.7518 - 10 \\ \operatorname{colog} 44.24 &= 8.3542 - 10 \\ \log \sin B_2 &= 9.8867 - 10 \end{aligned}$$

$$64$$

$$10)30(3$$

Then find C_2 (i. e., angle ACB_2) $= 95° 14'$. To find c_2 (i. e., side AB_2) use the law of sines again,

$$\frac{c_2}{44.24} = \frac{\sin 95° 14'}{\sin 34° 23'}$$

$$c_2 = 78.02$$

$$\begin{aligned} \log 44.24 &= 1.6458 \\ \log \sin 95° 14' &= 9.9982 - 10 \\ \operatorname{colog} \sin 34° 23' &= 0.2482 \\ \log c_2 &= 1.8922 \end{aligned}$$

$$21$$

$$6)10(2$$

To solve the triangle AB_1C, we first find $B_1 = 129° 37'$ being the supplement of B_2, and then the third angle $C_1 = 16° 00'$. To find c_1 (i. e., the side AB_1) use the law of sines,

$$\frac{c_1}{44.24} = \frac{\sin 16°}{\sin 34° 23'}$$

$$c_1 = 21.59$$

$$\begin{aligned} \log 44.24 &= 1.6458 \\ \log \sin 16° &= 9.4403 - 10 \\ \operatorname{colog} \sin 34° 23' &= 0.2482 \\ \log c_1 &= 1.3343 \end{aligned}$$

Check.

$$\begin{aligned} c_2 &= 78.02 \\ c_1 &= 21.59 \\ c_2 - c_1 &= 56.43 \end{aligned}$$

$$B_1B_2 = 2PB_2$$
$$= 2(44.24 \cos 50° 23')$$

$$\begin{aligned} \log 2 &= 0.3010 \\ \log 44.24 &= 1.6458 \\ \log \cos 50° 23' &= 9.8046 - 10 \\ \log B_1B_2 &= 1.7514 \end{aligned}$$

$$B_1B_2 = 56.41$$

Example 2. One angle of a triangle is 34° 23′, the side opposite is 60.35 and another side is 44.24. Solve.

There is only one solution as shown by constructing.

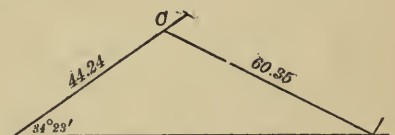


Fig. 56

$$\frac{44.24}{60.35} = \frac{\sin B}{\sin 34° 23'},$$

whence $B = 24° 27'$ and the third angle $C = 121° 10'$.

$$\frac{c}{60.35} = \frac{\sin 121° 10'}{\sin 34° 23'},$$

whence $c = 91.46$

EXERCISES

1. Two sides of a triangle are 17.16 and 14.15 and the angle opposite the latter is 42°. Find the other parts.

Ans. 125° 46′, 12° 14′, 4.483, or 54° 14′, 83° 46′, 21.02

2. In the triangle AGK, $A = 31° 14'$, $a = 54$, $g = 48.6$. Find the other parts. *Ans.* 27° 49′, 120° 57′, 89.3

3. A 50 ft. chord of a circle subtends an angle of 100° at the center. A triangle is to be inscribed in the larger segment having one side 40 ft. long. How long is the third side? How many solutions?

Ans. 65.22

4. If the triangle of Ex. 3 is to have one side 60 ft. long, how many solutions? How long is the third side. *Ans.* 18.88 or 58.25

105. Case III. Given two Sides and the included Angle. There is always one and only one solution. The third side can be found by the law of cosines and if the angles are not required, this is a convenient method of solution, especially if the given sides are not large.

Example 1. Two sides of a triangle are 2.1 and 3.5 and the included angle is 53° 8′. Find the third side.

$$x^2 = \overline{2.1}^2 + \overline{3.5}^2 - 2(2.1)(3.5)\cos 53° 8'$$
$$= 4.41 + 12.25 - 14.7 \times 0.6000 = 7.84,$$

whence $x = 2.8$.

If the other two angles as well as the third side are required, the two angles should be found by the law of tangents and then the third side can be found by the law of sines. Both these computations can be made by logarithms.

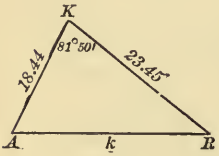


Fig. 57

Example 2. In the triangle ARK,

$a = 23.45$, $r = 18.44$, and $K = 81° 50'$.

Find the other parts.

By the law of tangents,

$$\frac{a + r}{a - r} = \frac{\tan \tfrac{1}{2}(A + R)}{\tan \tfrac{1}{2}(A - R)}.$$

The actual computation may be arranged as follows.

$a = 23.45$
$r = 18.44$
$a + r = 41.89$
$a - r = 5.01$

$180° 00'$
$K = 81° 50'$
$A + R = 98° 10'$
$\tfrac{1}{2}(A + R) = 49° 5'$

$$\frac{41.89}{5.01} = \frac{\tan 49° 5'}{\tan \tfrac{1}{2}(A - R)}$$

$\log 5.01 = 0.6998$
$\log \tan 49° 5' = 0.0621$
$\operatorname{colog} 41.89 = 8.3779 - 10$
$\log \tan \tfrac{1}{2}(A - R) = 9.1398 - 10$

$\tfrac{1}{2}(A - R) = 7° 51'$
$\tfrac{1}{2}(A + R) = 49° 5'$
$A = 56° 56'$ $R = 41° 14'$

CHECK.

$$\frac{23.45}{18.44} = \frac{\sin 56° 56'}{\sin 41° 14'}$$

$$\begin{array}{rl} \log 23.45 = & 1.3701 \\ \log \sin 41° 14' = & 9.8190 - 10 \\ \hline & 1.1891 \end{array} \qquad \begin{array}{rl} \log 18.44 = & 1.2658 \\ \log \sin 56° 56' = & 9.9233 - 10 \\ \hline & 1.1891 \end{array}$$

To compute k use the law of sines,

$$\frac{k}{23.45} = \frac{\sin 81° 50'}{\sin 56° 56'},$$

whence $k = 27.70$

EXERCISES

1. In the triangle ABC given $a = 52.8$, $b = 25.2$, $C = 124° 34'$; find the other parts. *Ans.* 38° 15′, 17° 11′, 70.2

2. Given $l = 131$, $m = 72$, $N = 39° 46'$, find n, L, M.
Ans. 88.57, 108° 54′, 31° 20′.

3. Given $u = 604$, $v = 291$, $W = 106° 19'$, find U, V, w.
Ans. 51° 32′, 22° 9′, 740.4

4. To find the distance between two objects A and B, separated by a swamp, a station C is selected so that $CA = 300$ ft., $CB = 277$ ft., and angle $ACB = 65° 47'$, can be measured. Compute AB.
Ans. 313.9

5. Two sides of a parallelogram are 23.47 and 62.38 and one angle is 71° 30′. Find its diagonals. *Ans.* 59.27 and 73.29

106. Case IV. Given the three Sides. There is one and only one solution, provided no side is greater than the sum of the other two.

The angles can be computed, in succession, by the law of cosines.

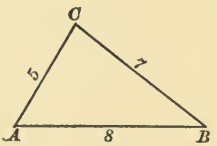


FIG. 58

EXAMPLE 1. The sides of a triangle are 5, 7, 8. Find the angles.

$$49 = 25 + 64 - 2 \times 5 \times 8 \cos A,$$

whence $\cos A = \frac{1}{2}$, $A = 60°$.

$$25 = 49 + 64 - 2 \times 7 \times 8 \cos B,$$
$$\cos B = \tfrac{11}{14} = 0.7857, \qquad B = 38° 13'.$$
$$64 = 25 + 49 - 2 \times 5 \times 7 \cos C,$$
$$\cos C = \tfrac{1}{7} = 0.1429, \qquad C = 81° 47'.$$

CHECK. $60° + 38° 13' + 81° 47' = 180° 00'$.

The law of cosines is not adapted to logarithms but can be transformed as follows. The three sides of a triangle ABC, being given, then

$$a^2 = b^2 + c^2 - 2bc \cos A,$$

whence

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}. \tag{41}$$

To adapt this to logarithmic computation, subtract each member from unity

$$1 - \cos A = 1 - \frac{b^2 + c^2 - a^2}{2bc} = \frac{2bc - b^2 - c^2 + a^2}{2bc}$$
$$= \frac{a^2 - (b - c)^2}{2bc}.$$

Hence we have

$$2 \sin^2 \tfrac{1}{2}A = 1 - \cos A = \frac{(a + b - c)(a - b + c)}{2bc}. \tag{42}$$

If we now set $a + b + c = 2s$, we have

$$a + b - c = 2(s - c),$$
$$a - b + c = 2(s - b).$$

Substituting these values in (42) we find

$$\sin^2 \tfrac{1}{2}A = \frac{(s - b)(s - c)}{bc}. \tag{43}$$

Similarly,

$$\sin^2 \tfrac{1}{2}B = \frac{(s - a)(s - c)}{ac}, \qquad \sin^2 \tfrac{1}{2}C = \frac{(s - a)(s - b)}{ab}.$$

Again, adding each member of (41) to unity,

$$1 + \cos A = 1 + \frac{b^2 + c^2 - a^2}{2bc} = \frac{(b + c)^2 - a^2}{2bc}$$
$$= \frac{(b + c + a)(b + c - a)}{2bc}.$$

Therefore,

$$2\cos^2 \tfrac{1}{2}A = 1 + \cos A = \frac{2s(s - a)}{bc},$$

whence

$$\cos^2 \tfrac{1}{2}A = \frac{s(s - a)}{bc}. \tag{44}$$

Similarly,

$$\cos^2 \tfrac{1}{2}B = \frac{s(s - b)}{ac}, \qquad \cos^2 \tfrac{1}{2}C = \frac{s(s - c)}{ab}.$$

Dividing $\sin^2 \frac{1}{2}A$ by $\cos^2 \frac{1}{2}A$, we have, by (43) and (44),

$$\tan^2 \tfrac{1}{2}A = (s - b)(s - c)/s(s - a)$$
$$= (s - a)(s - b)(s - c)/s(s - a)^2.$$

It follows that

$$\tan \tfrac{1}{2}A = \frac{1}{s - a}\sqrt{\frac{(s - a)(s - b)(s - c)}{s}}. \tag{45}$$

If we now set

$$r = \sqrt{(s - a)(s - b)(s - c)/s}, \tag{46}$$

the equation (45) becomes

$$\tan \tfrac{1}{2}A = \frac{r}{s - a}. \tag{47}$$

Similarly,

$$\tan \tfrac{1}{2}B = \frac{r}{s - b}, \qquad \tan \tfrac{1}{2}C = \frac{r}{s - c}.$$

It will be shown in § 107 that r is the radius of the circle inscribed in the given triangle.

EXAMPLE. The sides of a triangle are 77, 123, 130. Find the angles.

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} \qquad \tan \tfrac{1}{2}A = \frac{r}{s-a}$$

a	$= 77$
b	$= 123$
c	$= 130$
$2s$	$= 330$
s	$= 165$
$s - a$	$= 88$
$s - b$	$= 42$
$s - c$	$= 35$
CHECK	165

$\log (s-a)$	$= 1.9445$
$\log (s-b)$	$= 1.6232$
$\log (s-c)$	$= 1.5441$
colog s	$= 7.7825 - 10$
	$2)2.8943$
$\log r$	$= 1.4472$
$\log \tan \frac{1}{2}A$	$= 9.5027 - 10$
$\log \tan \frac{1}{2}B$	$= 9.8240 - 10$
$\log \tan \frac{1}{2}C$	$= 9.9031 - 10$
$\frac{1}{2}A$	$= 17° 39'$
$\frac{1}{2}B$	$= 33° 42'$
$\frac{1}{2}C$	$= 38° 40'$
CHECK	$90° 01'$

Therefore $A = 35° 18'$, $B = 67° 24'$, $C = 77° 20'$.
The sum of the half angles should check within 3′.

107. Area of a Triangle. It is shown in plane geometry that *the area of a triangle is equal to one half the product of any side and the perpendicular from the opposite vertex upon that side.*

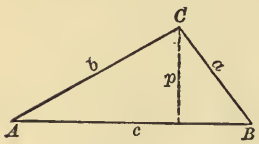


FIG. 59

If two sides and their included angle are given, say b, c, and A, then

$$p = b \sin A$$

and

$$\text{Area} = \tfrac{1}{2}bc \sin A, \tag{48}$$

whence, *the area of a triangle is equal to one half the product of any two sides and the sine of their included angle.*

If the three sides are given, a formula for the area can be deduced from (48) as follows. From (26), § 96, we have

$$\sin A = 2 \sin \tfrac{1}{2}A \cos \tfrac{1}{2}A$$

$$= 2 \frac{\sqrt{s(s-a)(s-b)(s-c)}}{bc},$$

by (43) and (44), § 106. It follows that

$$\text{Area} = \sqrt{s(s-a)(s-b)(s-c)}, \tag{49}$$

in which s denotes one half the perimeter.

Let r be the radius of the inscribed circle of the triangle whose sides are a, b, c. Then since the area of the triangle

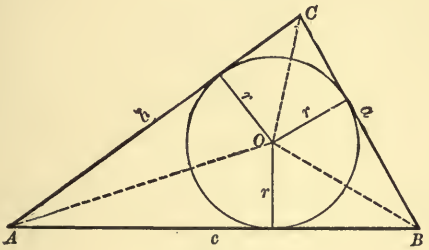


FIG. 60

ABC is equal to the sum of the areas of the triangles AOB, BOC, COA, we have,

$$\text{Area} = \tfrac{1}{2}cr + \tfrac{1}{2}ar + \tfrac{1}{2}br = rs. \tag{50}$$

Equating (49) and (50), and dividing through by s,

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}, \tag{51}$$

which proves that the r of § 106 is in fact the radius of the inscribed circle.

EXERCISES

1. Solve each of the following triangles.

(*a*) $a = 50$, $A = 65°$, $B = 40°$.

Ans. $C = 75°$, $b = 35.46$, $c = 53.29$

(*b*) $a = 30$, $b = 54$, $C = 46°$.

Ans. $A = 33° 6'$, $B = 100° 54'$, $c = 39.56$

(*c*) $a = 872.5$, $b = 632.7$, $C = 80°$.

Ans. $A = 60° 36'$, $B = 39° 24'$, $c = 986.2$

(*d*) $a = 120$, $b = 80$, $B = 35° 18'$.

Ans. $A = 60°$, $C = 84° 42'$, $c = 137.9$

(*e*) $a = 77.99$, $b = 83.39$, $C = 72° 15'$.

Ans. $A = 51° 15'$, $B = 56° 30'$, $c = 95.24$

2. Solve each of the following triangles.

	Given parts.		
(*a*)	$A = 21° 30'$,	$B = 75°$,	$a = 31.24$
(*b*)	$A = 62° 15'$,	$B = 48° 45'$	$b = 402.3$
(*c*)	$A = 53° 25'$,	$B = 70° 35'$,	$c = 6.031$
(*d*)	$a = 30$,	$b = 50$,	$A = 20°$.
(*e*)	$a = 25.8$,	$b = 40$,	$A = 40° 10'$.
(*f*)	$a = 37$,	$b = 25$,	$A = 37°$.
(*g*)	$a = 25.3$,	$b = 54$,	$A = 28°$.
(*h*)	$a = 42$,	$b = 42$,	$A = 56°$.
(*i*)	$a = 3$,	$b = 2$,	$C = 30°$.
(*j*)	$a = 640$,	$b = 800$,	$C = 48° 10'$.
(*k*)	$a = .0428$,	$c = .0832$,	$B = 58° 30'$.
(*l*)	$a = 12$,	$b = 16$,	$c = 22$.
(*m*)	$a = 6.02$,	$b = 4.82$,	$c = 8.12$

	Answers: Required parts		
(*a*)	$C = 83° 30'$,	$b = 82.32$,	$c = 84.68$
(*b*)	$C = 69°$,	$a = 473.4$,	$c = 499.4$
(*c*)	$C = 56°$,	$a = 5.841$,	$b = 6.861$
(*d*)	$C_1 = 125° 14'$,	$B_1 = 34° 46'$,	$c_1 = 71.63$
	$C_2 = 14° 46'$,	$B_2 = 145° 14'$,	$c_2 = 1.577$
(*e*)	$c = 30.57$	$C = 50°$,	$B = 90°$.
(*f*)	No solution		
(*g*)	No solution		
(*h*)	$B = 56°$,	$C = 68°$,	$c = 46.97$
(*i*)	$A = 111° 44'$,	$B = 38° 16'$,	$c = 2.403$
(*j*)	$A = 51° 58'$,	$B = 79° 52'$,	$c = 605.4$
(*k*)	$A = 30° 58'$,	$C = 90° 32'$,	$b = .0709$
(*l*)	$A = 32° 10'$,	$B = 45° 12'$,	$C = 102° 38'$.
(*m*)	$A = 47° 24'$,	$B = 36° 8'$,	$C = 96° 24'$.

3. Find the areas of each of the following triangles.

(*a*) Given $a = 40$, $b = 13$, $c = 37$. *Ans.* Area = 240.

(*b*) Given $a = 502$, $b = 62$, $c = 484$. *Ans.* Area = 14,590.

(*c*) Given $a = 27.2$, $b = 32.8$, $C = 65° 30'$. *Ans.* Area = 406.

(*d*) Given $a = 38$, $c = 61.2$, $B = 6\ ° 56'$. *Ans.* Area = 1,078.

4. Venus is nearer to the Sun than the Earth. Assume that the orbit of Venus is a circle with the Sun at its center. The distance from the Earth to the Sun is 92.9 millions of miles. What is the distance from Venus to the Sun if the greatest angular distance of Venus from the sun as seen from the Earth is 46° 20′? *Ans.* 67,200,000 mi.

5. On a clear day, twilight ceases when the sun has reached a position 18° below the horizon ($HAS = 18°$). Find the height AE of the atmosphere which is sufficiently dense to reflect the sun's rays. Take $OC = 4{,}000$ miles. The result must be diminished by 20% to allow for refraction. [Moritz]. *Ans.* 40 miles.

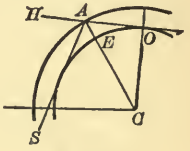


Fig. 61

6. The mean distances of the Earth and Mars from the sun are 92.9 and 141.5 millions of miles respectively. How far is Mars from the Earth when its angular distance from the sun is 28° 10′?

Ans. 21,280,000 mi.

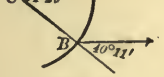


Fig. 62

7. From two points on the same meridian, the zenith distances of the moon are 35° 25′ and 40° 11′. The difference in latitude between the points of observation is 74° 26′. Find the distance of the moon from the earth, assuming the radius of the earth as 3,959 miles. [Moritz]

Ans. 239,000 miles, approximately.

8. A search light 20 feet above the edge of a tank is directed to a point on the surface of the water 40 feet from the edge. If the tank is 15 feet deep how far will be the illuminated spot on the floor of the tank from the edge, the index of refraction being 4/3? *Ans.* 62.5 ft.

9. A man whose eye is 6 feet above the edge of a tank 10 feet deep sees a coin in a direction making an angle of 34° with the surface of the water. If the index of refraction is 4/3, how far is the coin from the side of the tank? *Ans.* 16.83 ft.

10. Three forces of 12, 16, and 22 pounds in equilibrium can be represented by the 3 sides of a triangle taken in order. Find the angles which they make with each other.

Ans. 77° 22′, 134° 48′, 147° 50′.

11. A sharpshooter and an enemy are 220 feet apart and on the same side of a street 100 feet wide. Both are concealed by buildings. A bullet striking a building on the opposite side of the street at an angle x is deflected from the building at an angle y so that $3 \sin x = 4 \sin y$. Find x so that the sharpshooter may be able to hit the enemy.

Ans. 40° 6′.

12. A ship is going 15 miles per hour. How far to the side of a target 1 mile distant must the gunner aim if the shot travels 2000 ft. per second and the shot is fired when directly opposite?

Ans. 0° 38′ or 58 ft.

13. An aëroplane is observed from the base and from the top of a tower 40 feet high. The angles of elevation are found to be 10° 40′ and 9° 50′. Find the distance from the base to the plane and the height of the plane. *Ans.* 2713 ft., 502.4 ft.

14. To determine the distance of a hostile fort A from a place B, a line BC and the angles ABC and BCA were measured and found to be 1006.6 yd., 44°, and 70°, respectively. Find the distance AB.

Ans. 1,036 yd.

15. In order to find the distance between two objects, A and B, separated by a pond, a station C was chosen, and the distance $CA = 426$ yd., $CB = 322.4$ yd., together with the angle $ACB = 68° 42′$, were measured. Find the distance from A to B. *Ans.* 430.9 yd.

16. A surveyor wished to find the distance of an inaccessible point O from each of two points A and B, but had no instrument with which to measure angles. He measured $AA' = 150$ ft. in a straight line with OA, and $BB' = 250$ ft. in a straight line with OB. He then measured $AB = 279.5$ ft., $BA' = 315.8$ ft., $A'B' = 498.7$ ft. From these measurements find each of the distances AO and BO.

Ans. 152.3 ft., 319.7 ft.

17. Two stations, A and B, on opposite sides of a mountain, are both visible from a third station C. The distance $AC = 11.5$ mi., $BC = 9.4$ mi., and angle $ACB = 59° 30′$. Find the distance between A and B.

Ans. 10.5 mi.

CHAPTER VI

LAND SURVEYING

108. The Surveyor's Function. Land surveying consists in measuring distances and angles and marking corners and lines upon the ground, and in recording these measurements in field notes from which a map can be drawn and the area computed.

The original survey of a tract of land having been made and recorded, a surveyor may subsequently be called upon to find the corners, to relocate them if lost, to retrace the old boundaries, and to renew the corner posts and monuments if decayed or destroyed. This is called a *resurvey*.

A surveyor may make a resurvey of a tract of land in order to divide it by new lines and to map and compute the areas of the subdivisions.

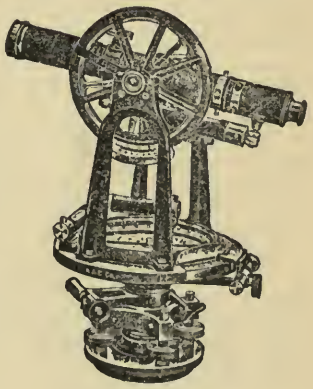


FIG. 63

109. Instruments. Distances on the ground are measured with the ***chain*** or ***tape.*** The land surveyor's chain is 66 feet (4 rods) long and is divided into 100 *links* each 7.92 inches long. The steel tape is usually 100 feet long, subdivided to hundredths of a foot.

Angles, horizontal or vertical, are usually measured with the ***transit.*** This is an instrument mounted on a tripod, and composed of the following parts: (*a*) the *telescope* provided with cross hairs to determine the line of sight, a sensitive spirit level, and a graduated circle on which the

angular turn of the telescope in the vertical plane is read; (*b*) the *alidade*, carrying the telescope, provided with spirit levels to bring its base into the horizontal plane and a large graduated circle on which is read the angular turn of the telescope in measuring horizontal angles; and (*c*) the *magnetic compass*.

110. Bearing of Lines. The direction of a line on the ground may be given by its ***bearing;*** this is the angle between the line and the meridian through one end of it. For example, a line bearing N 26° E is one which makes an angle of 26° on the east side of north; one bearing S 85° W makes an angle of 85° on the west side of south. The bearing of a line which is run by the transit is read off on the compass circle but is subject to a correction depending upon the time and place since the magnetic needle does not point due north at all times and places.

111. Government Surveys. In government surveys of the public lands, a north and south line called a ***principal meridian*** is first accurately laid out and marked by permanent monuments. From a convenient point on the principal meridian a ***base line*** is run east and west and carefully marked. North and south lines, called ***range lines,*** are then run from points six miles apart on the base line. Then ***township lines*** six miles apart are run east and west from the principal meridian.

The land is thus divided into ***townships*** six miles square. A tier of townships running north and south is called a ***range.*** Ranges are numbered consecutively east and west from the principal meridian. Townships are numbered north and south from the base line.

In deeds and records a township is located, not by the county, but as " Township No. — north (or south) of a certain base line and in range No. — east (or west) of a certain principal meridian. Townships are divided into thirty-six *sections* each one mile square containing 640 acres, and are numbered from

1 to 36 as shown in Fig. 64. The sections are often subdivided into halves, quarters, eighths, etc., as illustrated in Fig. 65.

6	5	4	3	2	1
7	8	9	10	11	12
18	17	16	15	14	13
19	20	21	22	23	24
30	29	28	27	26	25
31	32	33	34	35	36

Fig. 64

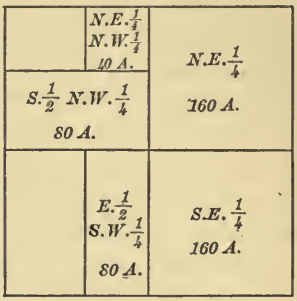


Fig. 65

The first principal meridian runs north from the junction of the Ohio and Big Miami rivers on the boundary between Ohio and Indiana. The second coincides with 86° 28′ of longitude west of Greenwich running north from the Ohio river near the towns of English, Bedford, Lebanon, Culver, Walkerton, and Warwick, Indiana. The surveys in Indiana (with the exception of certain lands in the southeast corner) are governed by this second principal meridian and a base line in latitude 38° 28′ 20″ crossing this meridian about 5 miles south of Paoli, in Orange County.* Thus a certain parcel of land is described in the Indiana records as "E ½ of NW ¼ of Section nineteen (19), Township twenty-three (23) N, Range four (4) W."

The surveys extending east from one meridian will not generally close with those extending west from the preceding meridian; the same is true of the ranges of townships extending north

*The first six principal meridians are designated by number; some twenty-odd others by name. E. g., the Mount Diablo meridian, 120° 54′ 48″ W, which governs surveys in California and Nevada. The first six base lines are neither numbered nor named but all subsequent ones are named. The locations of all the principal meridians and base lines is given in the *Manual of Instructions for the Survey of the Public Lands* issued from time to time by the General Land Office, Washington. D. C. For details and a historical sketch see also, Pence and Ketchum, *Surveying Manual.*

and south from the base lines. These circumstances and the presence of rivers and lakes give rise to fractional townships and sections.

112. **Corners.** In an original survey one of the most important of the surveyor's duties is the marking of corners in such a manner as to perpetuate their location as long as possible. The *Manual of Instructions* (see 1894 edition, p. 44) says, "If the corners be not perpetuated in a permanent and workmanlike manner, the *principal object* of the surveying operations will not have been attained."

The Instructions prescribe in detail the kind of monument and the mark to be put upon it to establish each of the various kinds of corners that are located in the government surveys. Wooden posts and stakes, stones, trees, and mounds of earth are used. Witness trees or witness points are trees or other objects located near the corner, suitably marked and described in the field notes to make easy a subsequent relocation of the corner.

If called upon to make a resurvey of land that was originally laid out under the direction of the General Land Office, the surveyor will do well to make a careful study of the instructions concerning corners that were in force when the original survey was made, as the practice has varied somewhat from time to time.

113. **Judicial Functions of Surveyors.*** Many years have elapsed since the greater part of the government surveys were made and in many cases the original corner marks have entirely disappeared. The first settlers and original owners often failed to fix their lines accurately while the monuments remained, and the subsequent owners have no first hand knowledge of their location. When in such cases a surveyor is called upon to

* This topic is based upon a paper of the same title by Justice Cooley of the Michigan Supreme Court, published in the Michigan Engineer's Annual for 1880–81, pp. 18–25.

make a resurvey, it is his duty to find if possible where the original corners and boundary lines were, and not at all where they ought to have been. However erroneous the original survey may have been, the monuments that were set must nevertheless govern, for the parties concerned have bought with reference to these monuments and are entitled only to what is contained within the original lines.

If the original monument and all the witness trees and other identification marks mentioned in the field notes of the original survey have disappeared, the corner is *lost* and it is the duty of the surveyor to relocate it in the light of all the evidence in the case, including the testimony of persons familiar with the premises, existing fences, ditches, etc., at the point where this evidence shows it most probably was.

In making a resurvey the surveyor has no authority to settle disputed points; if the disputing parties do not agree to accept his decision, the question must be settled in the courts. In a controversy between adjacent owners over the location of corners and division lines, it is well established in law that a supposed boundary line long accepted and acquiesced in by both parties is better evidence of where the real line should be than any survey made after the original monuments have disappeared. It is common belief that boundary lines do not become fixed by acquiescence in less than 21 years, but there is no particular time that must elapse to establish boundary lines between private owners where it appears that they have accepted a particular line as their boundary and all concerned have claimed and occupied up to it.

114. **Measuring on Level Ground.** The line to be measured is first ranged out and marked with range poles or its direction is determined by the line of sight of the transit set on the line. The leader sticks a pin at the starting point, takes ten in his hand and steps forward on the line dragging the

chain behind him. At a signal from the follower, given just before the full length has been drawn out, he turns, aligns, and levels the chain, stretches it to the proper tension, and, while the follower holds the rear end at the starting point, sticks a pin at the forward end on the line determined by the follower and a range pole or by the transitman. At a signal from the leader the follower pulls his pin and both move forward on the line another chain's length and set the next pin. This process is repeated until the leader has set his tenth pin, when the follower goes forward, counting his pins as he goes and, if there are ten, hands them to the leader who also counts them. The count of tallies is kept by both. When the end of the line is reached the follower walks forward and reads the fraction of the chain at the pin and notes the number of pins in his hand to determine the distance from the last tally point which is recorded in the field notes.

115. Measuring on Slopes. The horizontal distance which is required can be found on slopes by leveling the chain and plumbing down from the end off ground. On steep slopes only a part of the chain can be used as at A and B in Fig. 66. The part used should be a multiple of ten links and great caution must be used by both leader and follower to avoid mistakes and confusion in the count of pins.

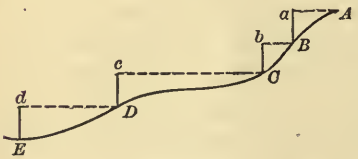


FIG. 66

116. Offsets. In case measurements cannot be made on the desired line on account of a fence, hedge, pond or other obstacle, a perpendicular to the line, called an ***offset***, is measured, sufficiently long to avoid the obstruction and the measurements are

made on an auxiliary line parallel to the required line. Stakes may then be set on the required line by offsets from the auxiliary line. See Fig. 67.

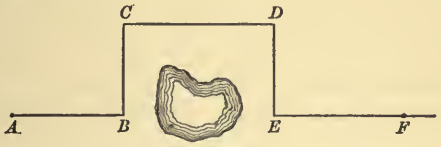


Fig. 67

From any point on a line a right angle (or any other required angle) can be laid off with the transit. An angle of 90° or 60° can be laid off in a clear space with chain or tape and pins as shown in Fig. 68, from the facts that (1) a triangle whose sides are to each other as 3 : 4 : 5 has a right angle opposite the longest side; and (2) an equilateral triangle has three 60° angles.

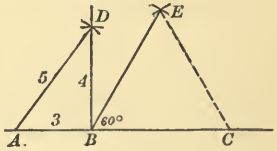


Fig. 68

117. Passing Obstacles. An obstacle in the line can be passed and the line prolonged beyond it by means of perpendicular offsets as shown in Fig. 67, if the nature of the locality makes it convenient.

The same result can be accomplished by a triangle as shown in Fig. 69. The angle HAB, the distance AB, the angle KBC, are measured; then the distance BC and the angle MCD are computed; the distance BC is measured off and the point C is located and the angle at C is turned off and the direction CD established; AC is computed and the point D is taken a whole number of chains from A. The angles at A and B and the

Fig. 69

distance AB are arbitrary and may be taken so as to avoid difficulties of the surroundings. If the circumstances permit the angle HAB may be made 60°, and angle $KBC = 120°$; then the triangle ABC will be equilateral and computations will be avoided.

118. Random Lines. When it is desired to mark out a long line, such as AB, Fig. 70, whose end points are established but

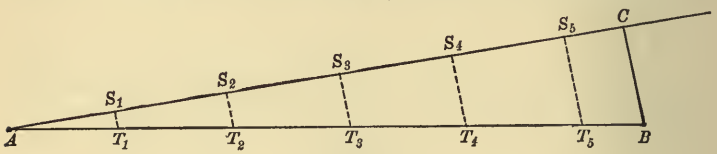


FIG. 70

are invisible each from the other, a line AC, called a ***random line***, is run as nearly in the direction of AB as can be determined and stakes S_1, S_2, S_3, etc., are set at regular measured distances. On coming out near B a perpendicular is let fall from B to AC precisely locating the point C. The lengths of the offsets S_1T_1, S_2T_2, S_3T_3, etc., all perpendicular to AC, can be computed and on retracing CA, stakes can be set at T_5, T_4, T_3, etc., on the desired line AB. For example, if the stakes on AC are 12 chains apart, if $S_5C = 6.46$ chains, and if $BC = 54$ links, then the offset, in links, at any stake S, is found by multiplying its distance AS, in chains, from A, by the ratio $54/66.46 = 0.8125$. Thus the offset $S_4T_4 = 48 \times 0.8125 = 39$. It is left to the student to show that AB is longer than AC by less than 1/4 a link and that the stakes on AB are practically 12 chains apart.

119. Computing Areas. If the boundaries of a tract are all straight lines, i. e., if its perimeter is a polygon, the area can be computed by dividing it into triangles, or into rectangles and triangles, provided enough measurements are made so that the required dimensions of each part are known or can be com-

puted. It is customary to measure more lines on the ground than is theoretically necessary in order to check the computations. These extra measurements are called ***proof lines*** in the field notes.

120. Irregular Areas by Offsets. When one side of a field is not straight as occurs if the boundary is a stream or curved road, a line may be run cutting off the irregular part and leaving the remainder of the field in a shape whose area is easily computed; as AD in Fig. 71. Stakes are set at regular measured intervals on AD and the offsets AB, S_1T_1, S_2T_2, S_3T_3, etc., are measured. The area can be approximated by considering each of the strips to be a trapezoid. On computing and adding we are led to the following rule.

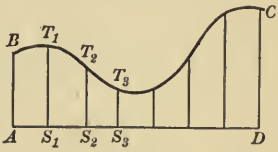


Fig. 71

Rule: *From the sum of all the offsets subtract half the sum of the extreme ones and multiply the remainder by the common distance between them.*

121. Areas by Rectangular Coördinates. If the irregular side of the field is a broken line or if the nature of the place makes it inconvenient to measure the offsets at regular intervals the area can be found by measuring the rectangular coördinates of the points A, B, C, D, E, Fig. 72, referred to the axes OX and OY.

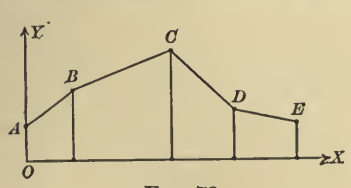


Fig. 72

Let the coördinates of A, B, C, $\cdots$ be $(0, y_0)$, (x_1, y_1), (x_2, y_2), $\cdots$ respectively. Then the sum of the areas of the trapezoids is

$$(1)\quad \tfrac{1}{2}[x_1(y_0 + y_1) + (x_2 - x_1)(y_1 + y_2) + \cdots + (x_n - x_{n-1})(y_{n-1} + y_n)],$$

where n is the number of trapezoids. On combining terms this reduces to

$$(2)\quad \tfrac{1}{2}[\{x_1(y_0 - y_2) + x_2(y_1 - y_3) + \cdots + x_{n-1}(y_{n-2} - y_n)\} + x_n(y_{n-1} + y_n)].$$

Hence we have the following rule.

RULE: *From each ordinate subtract the second succeeding ordinate and multiply the remainder by the abscissa of the intermediate point; also multiply the sum of the last two ordinates by the last abscissa; and divide the algebraic sum of the products by two.*

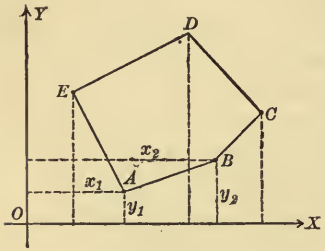


FIG. 73

If the coördinates of the vertices of a closed polygon are known its area can be computed as follows. Consider the convex pentagon shown in Fig. 73. The area included may be found by adding the trapezoids under the sides ED and DC and subtracting those under the other three sides; this gives

$$(3)\quad \tfrac{1}{2}[(x_4 - x_5)(y_4 + y_5) + (x_3 - x_4)(y_3 + y_4) - (x_3 - x_2)(y_3 + y_2) - (x_2 - x_1)(y_2 + y_1) - (x_1 - x_5)(y_1 + y_5)].$$

Combining like terms, we find that this reduces to either

$$(4)\quad \tfrac{1}{2}[x_1(y_2 - y_5) + x_2(y_3 - y_1) + x_3(y_4 - y_2) + x_4(y_5 - y_3) + x_5(y_1 - y_4)],$$

or

$$(5)\quad \tfrac{1}{2}[y_1(x_5 - x_2) + y_2(x_1 - x_3) + y_3(x_2 - x_4) + y_4(x_3 - x_5) + y_5(x_4 - x_1)].$$

These formulas are easily extended to convex polygons of any number of sides and prove the following rule.

Multiply each abscissa by the difference of its adjacent ordinates, always making the subtractions in the same sense around the perimeter, and take one-half the algebraic sum of the products.

The result will be the same (except as to sign) if in this rule the words *abscissa* and *ordinate* be interchanged.

EXERCISES

1. Find the area of a field in the form of a right triangle.

(*a*) Base = 31.28 ch., Altitude = 16.25 ch. *Ans.* 25.42 A.

(*b*) Base = 28.46 ch., Altitude = 38.65 ch. *Ans.* 55.00 A.

2. Find the area of a triangular field,

(*a*) whose three sides are 24.50, 10.40, and 21.70 ch.

(*b*) having two sides 35.60, 23.70 ch., and their included angle 42° 30′.

Ans. (*a*) 11.27 A. (*b*) 28.50 A.

3. How many acres in a rectangular field whose dimensions are 17.44 and 32.65 ch. *Ans.* 56.94 A.

4. One side of a 200 acre rectangular field is 33.60 chains. Find the other side. *Ans.* 62.50 ch.

5. What is the length of one side of a square field which contains 36 acres? *Ans.* 18.97 ch.

6. The diagonals of a four-sided field measure 21.40 and 24.50 ch., and they cross at an angle of 74° 40′. Find the area. *Ans.* 25.28 A.

7. One diagonal of a quadrangle runs N. 36° 20′ E. 22.40 ch., and the other S. 69° 30′ E. Find its area. *Ans.* 25.22 A.

8. Find the areas of the fields whose boundaries are given.

(*a*)

Station.	Bearing.	Distance.
A	North	9.14 ch.
B	S. 73° 25′ E.	8.27
C	S. 28° 15′ E.	10.04
D	N. 80° 45′ W.	12.84½

(*b*)

Station.	Bearing.	Distance.
P	West	19.66 ch.
Q	North	13.77
R	N. 64° 15′ E.	16.66
S	S. 12° 30′ E.	21.51

Ans. (*a*) 8.74 A. (*b*) 30.97 A.

9. Find the areas of the fields whose boundaries are given.

(*a*)

Station.	Bearing.	Distance.
A	N. 25° 30′ E.	10.50 ch.
B	N. 76° 50′ E.	7.00
C	S. 19° 30′ E.	7.92
D	S. 53° 34′ W.	11.90
E	N. 64° 30′ W.	4.20

(*b*)

Station.	Bearing.	Distance.
1..	N. 12° 46′ W.	6.80 ch.
2..	N. 49° 10′ E.	2.40
3..	S. 40° 50′ E.	6.00
4..	S. 10° 30′ W.	4.00
5..	N. 85° 50′ W	4.52½

Ans. (*a*) 10.09 A. (*b*) 3.30 A.

10. The coördinates, in chains, of the vertices of a broken line are:

Vertex.	A.	B.	C.	D.	E.	F.
x	0.00	2.95	1.10	0.60	2.20	1.80
y	10.00	8.12	7.25	5.00	4.50	0.00

Find the area included by the broken line and the axes. *Ans.* 2.36 A.

11. The coördinates, in chains, of the corners of a field are:

Vertex.	1.	2.	3.	4.	5.	6.
x	0.00	7.00	12.50	18.00	15.00	10.00
y	6.00	12.00	20.00	15.00	8.25	0.00

Make a plot and find the area. *Ans.* 16.175 A.

12. Starting on the bank of a river a line is run across a bend 20.00 ch., to the bank again. Offsets are measured every two chains as follows: 1.61, 2.27, 1.96, 4.23, 3.70, 2.92, 3.26, 2.50, 1.25 ch. Make a plot of the land between the line and the river and find the area.

Ans. 4.74 A.

13. Find the measurements so as to run a line from the vertex A of a triangle ABC to a point D on the side BC = 8.75 ch., so as to cut off 2/5 of the area next to B. *Ans.* BD = 3.50 ch.

14. Find the measurements so as to run a line through a point E on BC of the triangle of Ex. 13, parallel to AB so as to cut off 2/5 of the area in the trapezoid. *Ans.* CE = 6.78 ch.

15. Two lines meet at P. PA bears S. 65° 30′ E., PB bears N. 78° 15′ E. Determine measurements to run a line perpendicular to PA so as to cut off five acres. *Ans.* Base $= 10\sqrt{\text{ctn } 36° 15'} = 11.68$ ch.

16. A triangular field contains 6 A. Show how to find on the plot a point inside the triangle from which lines drawn to the vertices will divide it into three triangular fields of 1, 2, and 3 A., and so that the smallest and largest shall be adjacent respectively to the smallest and largest sides of the field.

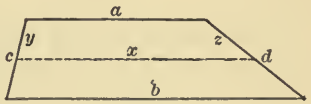


Fig. 74

17. If the bases of a trapezoid are a and b, $a < b$, and the slant sides are c and d, as in Fig. 74, determine measurements to run a line parallel to the bases to cut off, adjacent to the shorter base a, a fraction f, of the whole area.

$$Ans.\ x = \sqrt{a^2 + f(b^2 - a^2)},\quad y = c\,\frac{x - a}{b - a},\quad z = d\,\frac{x - a}{b - a}.$$

18. Given $a = 20$, $b = 30$, $c = 54.40$ ch., determine x and y to cut off $\frac{1}{3}$ the area, Fig. 74. *Ans.* $x = 23.80$, $y = 20.69$ ch.

19. In a four sided field $ABCD$, AB runs S. 8.40 ch., BC, E. 9.24 ch., and CD, N. 5.68 ch.

(*a*) Run a north and south line so as to divide it into two parts whose areas shall be to each other as 2 : 3 with the smaller on the east.

(*b*) Run a north and south line so as to cut off 3 A. on the west.

Ans. (*a*) 4.14 ch. from the east; (*b*) 5.40 ch. from the east.

20. A tract of land $ABCD$, lies between two converging streets as shown in Fig. 75. $AB = 1980$ ft., $AC = 590$ ft., $BD = 1380$ ft. Determine the measurements for running lines PQ, RS, etc., perpendicular to AB, so as to divide the tract into ten lots of equal area.

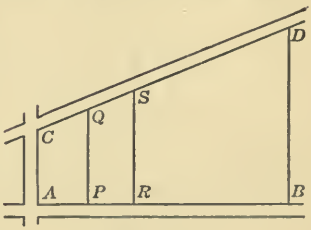


Fig. 75

[Hint. Use the method of Ex. 17 to find PQ and AP. Or otherwise, find the tangent of the angle between the streets AB and CD; find the

area of $CAPQ$ in terms of $x(= AP)$; this leads to a quadratic equation in x. Find the positive root.]

Ans. $AP = 300.11$ ft. $PQ = 709.74$ ft. Area $CAPQ = 4.477$ A.

21. From the notes in Ex. 8 (*b*), make a plot and (*a*) run a line from S to a point M on PQ so as to divide the field into two parts of equal areas, (*b*) run a line from R to a point N on SP so as to cut off 10 acres in the triangle.

22. From the accompanying notes from a farm survey compute the lengths of the first, second, and fourth sides.

Corner.	Bearing.	Distance.
1.....	N. 75° 30′ W.	
2.....	N. 5° 15′ E.	
3.....	S. 68° 10′ E.	30.08 ch.
4.....	S. 23° E.	
Area = 139.84 acres		

[Hint. Produce the second and fourth sides to form a triangle.]

Ans. 51.38, 36.56, 40.16.

23. Suppose the lengths of the first and fourth sides of the field in Ex. 9 (*a*) are unknown. Compute them from the other data if the area is 10.094 acres.

24. It is desired to mark out and measure a line PQ. A random line PR is run and stakes are set on it every 100 ft. The perpendicular from Q upon PR is 48.82 ft. long and meets it at R, 22.18 ft. beyond the 42nd stake. Compute the offsets for setting the stakes over on PQ, their distance apart, and the length of PQ.

25. To prolong a line AB past an obstacle O, a right turn 40° is made at B, 400 ft. is measured to C, and a left turn of 116° is made. Compute the distance to D on AB produced through O and the right turn which must be made at D. How far from D should hundred foot stakes be resumed?

CHAPTER VII

STATICS

122. **Statics.** *Statics* treats of bodies at rest and of bodies whose motion is not changing in direction or in speed. A body whose motion is not changing is said to be in ***equilibrium.*** The chief problem of statics is to find the conditions of equilibrium.

123. **Mass.** The weight, W, of a body is not constant. For instance a body weighs less on a mountain top than at sea level. Also the acceleration, g, due to gravity is not constant. It likewise is less on a mountain top than at sea level. An increase in acceleration is accompanied by a proportional increase in weight. But the ratio W/g is constant. The constant number represented by this ratio is called the ***mass*** of the body. A unit of mass is the ***gram***, and is 1/1000 of the mass of a certain piece of platinum which is preserved at Paris. Another unit of mass is the ***avoirdupois pound.*** One thousand grams is equal to 2.20462125 lbs. The mass of any body is then the number expressing the ratio of its weight to the weight of a unit of mass. The weight is to be determined by means of a spring balance.

124. **Momentum.** When a given mass is in motion, we require to know not only the magnitude of the mass, but also its velocity. *The product of the mass of a body and its velocity is called its* ***momentum.***

125. **Force.** If a body possesses a certain amount of momentum, it is impossible for it to alter its motion in any manner unless acted upon by some other body which pushes or pulls it.

Force is that which tends to produce a change of motion in a body on which it acts. This change of motion is proportional to the force and takes place in the direction of the straight line in

which the force acts. Thus, to increase the speed of an automobile, the driving force must be increased. The greater the force, the greater the rate of inercase in the speed.

This illustrates the fact that forces arc of different magnitudes.

If a motionless croquet ball is struck, its subsequent motion depends upon the direction of the stroke. This illustrates the fact that forces have different lines of action. If a billiard ball is struck, the motion of the ball depends upon the point at which the cue struck the ball. This illustrates the fact that we must take into account the point of application of the force.

A force is said to be completely determined if we know (*a*) its ***magnitude;*** (*b*) its ***line of action;*** (*c*) ***its direction along the line of action;*** (*d*) ***its point of application.***

In practice forces are never applied at a point. The force is applied over an area such as the pressure of a thumb on the head of a tack or the pressure of a book on a table. A foree may act throughout an entire volume as is the case with attraction. These forces are called *distributed forces.* In praetice we oftcn consider the forces which applied at a point would produce the same effect as the given distributed forces. Such forces are termed *concentrated forces.*

126. **Unit of Force.** The unit of force is sometimes taken as the weight of a unit mass. This unit of force is not constant. It changes both with altitude and with latitude. These changes are small but for scientific purposes cannot be neglected. To obtain a constant unit it is sufficient to make the following definition:

The unit of force is the weight of a unit of mass at a fixed place, say at London, Paris, or Washington.

127. **Graphic Representation of Forces.** A forcc P is completely determined if we know its magnitude, its line of action, its direction along this line, and its point of application. It

follows that a force can be completely represented by anything which possesses these attributes. It can, for example, be represented by a directed segment of a straight line. For we may let any point O, Fig. 76, represent the point of application. From O draw any line segment OA the number of units in whose length is the same as the number of units in the given force. The length of the segment represents then the magnitude of the force. The line of which OA is a part represents the line of action of the force. We can represent the direction along the line by an arrowhead placed on OA.

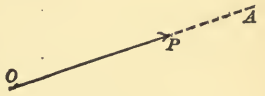


FIG. 76

128. Composition of Forces.—Parallelogram of Forces. If two or more forces act in the same straight line and in the same direction, their *resultant*, or sum, is obtained by adding the numbers representing the magnitudes of the forces.

If the forces act in the same straight line but in opposite directions, the resultant is equal to their difference, that is to their algebraic sum.

When the forces do not act in the same straight line the *total* or *resultant force* is found by means of a rule called the ***parallelogram of forces:*** *If two forces not in the same straight line are represented in direction and in magnitude by two adjacent sides of a parallelogram, the single force which would produce the same effect as the two given forces is represented in direction and in magnitude by that diagonal of the parallelogram which passes through the same vertex as the two given forces.*

In Fig. 77, the forces F_1 and F_2 are represented by the lines AB and AC, respectively. Their resultant R is represented by AD. The magnitude of the resultant is given by the equation

$$R = \sqrt{F_1^2 + F_2^2 + 2F_1F_2 \cos \theta}, \tag{1}$$

where θ stands for the angle BAC.

It will be noted that BD, being parallel and equal to AC, represents the magnitude and the direction (but not line of

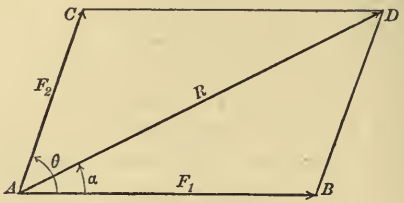


FIG. 77

action) of the force F_2. If we let α equal the angle BAD, we have, from the triangle BAD

$$\frac{R}{\sin\theta} = \frac{F_2}{\sin\alpha}, \tag{2}$$

whence

$$\sin\alpha = \frac{F_2 \sin\theta}{R}. \tag{3}$$

The direction α of the resultant force may be found from this equation. Thus R is completely determined.

When $\theta = 90°$, equation (1) reduces to

$$R = \sqrt{F_1^2 + F_2^2}. \tag{4}$$

We also have

$$\sin\alpha = \frac{F_2}{R}, \qquad \cos\alpha = \frac{F_1}{R}. \tag{5}$$

Two forces which have a given force for their resultant are called the ***components*** of this force. Thus F_1 and F_2 are components of R. The process of finding the resultant of any number of forces is known as the ***composition of forces.*** The process of finding the components of a given force is called the ***resolution of forces.*** Two systems of forces acting on a particle and having the same resultant are said to be ***equivalent.***

129. Rectangular Components of a Force. Frequently, it is desired to resolve a force into components which are, respectively, parallel and perpendicular to a given line. Such components are called *rectangular components.* In this case the magnitudes of the components may be found by the solution of equations (5), or directly from a figure. See Fig. 78. Thus we find

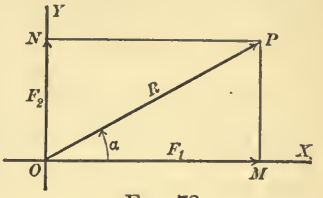


Fig. 78.

$$(6) \qquad F_1 = R\cos\alpha, \qquad F_2 = R\sin\alpha.$$

These formulas give F_1 and F_2 as the rectangular components of R.

Similarly the component of any given force along any given line is equal to the magnitude of the force multiplied by the cosine of the angle between the line and the force.

EXERCISES

1. Given $F_1 = 48.7$, $F_2 = 69.8$, $\theta = 65°\ 20'$, find R and α.
2. Given $F_1 = 20.3$, $F_2 = 60.2$, $\theta = 135°\ 10'$; find R and α.
3. Given $F_1 = 60.3$, $F_2 = 30.2$, $\theta = 90°$, find R and α.
4. Given $F_1 = 26.7$, $F_2 = 45\ 7$, $\theta = 60°$; find R and α.
5. $R = 140$, $\alpha = 15°$; find F_1 and F_2.
   Ans. $F_1 = 135.2$, $F_2 = 36.2$
6. $R = 125$, $\alpha = 25°$; find F_1 and F_2.
   Ans. $F_1 = 113.3$, $F_2 = 52.8$
7. $R = 325$, $\alpha = 35°$; find F_1 and F_2.
   Ans. $F_1 = 266.2$, $F_2 = 186\ 4$
8. $R = 600$, $\alpha = 55°$; find F_1 and F_2.
   Ans. $F_1 = 344.1$, $F_2 = 491.5$
9. A particle is acted upon by two forces, of 8 and 10 pounds respectively, making an angle of 30° with each other. Find the magnitude of the resultant. *Ans.* 17.39

10. A boat is being towed by two ropes making an angle of 60° with each other. The pull on one rope is 500 pounds, the pull on the other is 300 pounds. In what direction will the boat tend to move? What single force would produce the same result? [Miller-Lilly]

Ans. 21° 47′ with force of 500 lbs.; 700 lbs.

11. Let a raft move in a straight line down stream with a uniform speed of 2 feet per second; suppose a man upon the raft walks at a uniform speed of 4 feet per second in a direction making an angle of 60° with the direction of movement of the raft. Find the speed and direction of the man relative to the earth.

Ans. $\sqrt{28}$ ft. per sec. at an angle of 40° 54′ with direction of raft.

12. A river one mile wide flows at a rate of 2.3 miles per hour. A man, who in still water can row 4.2 miles per hour, desires to cross to a point directly opposite. Find in what direction he must row and how long he will be in crossing.

Ans. Upstream at an angle of 56° 48′ with direction of stream; 17 minutes approximately.

13. A man in a house observes rain drops falling with a speed of 32 feet per second. The direction of descent makes an angle of 30° with the vertical. Find the velocity of the wind.

Ans. 18.5 ft. per sec.

14. A motor boat points directly across a river which flows at the rate of 3.5 miles per hour; the boat has a speed in still water of 10 miles per hour. Find the speed of the boat and the direction of its motion.

Ans. 10.59 mi. per hr., 70° 43′ with direction of stream.

15. From a railway train going 40 mi. per hour a bullet is fired 2,000 ft. per second at an angle of 65° with the track ahead. Find its speed and direction.

130. Triangle of Forces. It will be seen at once on referring to Fig. 77 that the sum or resultant of the two forces F_1 and F_2 could be obtained more easily by drawing a triangle ABD, as in Fig. 79; when applied to find the resultant of two forces the triangle ABD is called the ***triangle of forces.***

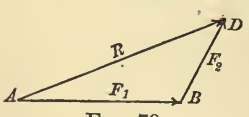


Fig. 79

Referring again to Fig. 79, it is evident that if a force equal

and opposite to the resultant R were applied at A, this force and the forces F_1 and F_2 would balance, and the point A would be in equilibrium. Another way of stating the proposition would be as follows.

If three concurrent forces are in equilibrium, their magnitudes are proportional to the three sides of a triangle whose sides, taken in order, are parallel to the directions of the given forces. Conversely, if the magnitudes of three concurrent forces are proportional to the three sides of a triangle and their directions are parallel to the sides taken in order, these forces will be in equilibrium.

EXERCISES

1. Draw a triangle ABC whose sides BC, CA, AB are 7, 9, 11 units long. If ABC is a triangle for three forces in equilibrium at a point P, and if the force corresponding to the side BC is a force of 21 lbs., show in a diagram how the forces act, and find the magnitude of the other two forces. *Ans.* 27, 33.

2. Draw two lines AB and AC containing an angle of 120°, and suppose a force of 7 units to act from A to B and a force of 10 units from A to C. Find by construction the resultant of the forces, and the number of degrees in the angle its direction makes with AB.

Ans. $\sqrt{79}$; 77°, approximately.

3. Draw an equilateral triangle ABC, and produce BC to D, making CD equal to BC. Suppose that BD is a rod (without weight) kept at rest by forces acting along the lines AB, AC, AD. Given that the force acting at B is one of 10 units acting from A to B, find by construction (or otherwise) the other two forces, and specify them completely.

4. Find the resultant of two velocities of 9 and 7 ft. per second acting at a point at an angle of 120°. *Ans.* $\sqrt{67}$.

5. Find the magnitude and direction of the resultant of two velocities of 5 and 4 ft. per second acting at a point at an angle of 45°.

Ans. 8.32; 19° 52′.

6. A certain clothes line which is capable of withstanding a pull of 300 pounds, is attached to the ends A and B of two posts 40 feet apart,

A and B being in the same horizontal line. When the rope is held taut by a weight W, attached to the middle point, C, of the line, C is four feet below the horizontal line AB. Find the weight of the heaviest boy it will support without breaking. [MILLER-LILLY]

Ans. 117.7, lbs.

7. A street lamp weighing 100 pounds is supported by means of a pulley which runs smoothly on a cable supported at A and B, on opposite sides of the street. If A is 10 feet above B, and the street 60 feet wide, and the cable 75 feet long, find the point on the cable where the pulley rests, and the tension in the cable. [MILLER-LILLY]

8. A particle of weight W lies on a smooth plane which makes an angle α with the horizon. Show that $P = W \sin \alpha$, $R = W \cos \alpha$, where P is the force acting along the plane to keep the particle from slipping and R is the reaction of the plane.

131. The Simple Crane. One of the most useful applications of the triangle of forces is the case of an ordinary crane. It has a fixed upright member AB called the *crane post*, a member AC called the *jib*, and a *tie-rod* BC. A weight W suspended

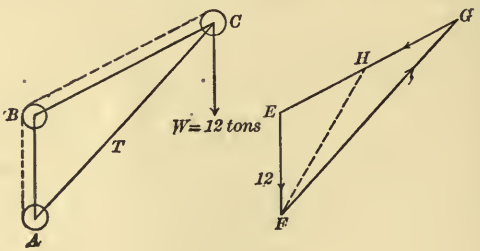


FIG. 80

rigidly at C is kept in position by three forces in equilibrium. These forces are (*a*) the weight W, (*b*) the pull in the tie-rod, and (*c*) the thrust in the jib. To determine their magnitudes construct to scale a force triangle EFG. Draw EF parallel to the line of action of the weight W and equal to W in magnitude. From F draw FG parallel to the jib and from E draw

EG parallel to the tie-rod. The lengths of EG and FG to the same scale on which EF was drawn represent the thrust in the jib and the pull in the tie-rod. The directions of the forces acting along the tie-rod and jib are given by following around the triangle in order from E to F to G to E.

When a crane is used to raise or lower a weight, the weight is held by a rope passing over a pulley at C. The tension of the rope must now be taken into account.

Suppose a chain or rope supporting the weight is made to pass over a pulley at C, and is then led on to a drum at A round which the rope or chain is coiled. The pull in the rope and tie-rod together is the same as before and is represented by EG. The tension in the rope is the same on each side of the pulley. Therefore if we mark off on EG a distance HE equal to EF, this distance will represent the pull in the rope, thus leaving GH to represent the pull in the tie-rod.

EXERCISES

Find the pull in the tie-rod and the thrust in the jib of a crane when the dimensions and weight are as given below. (Weight suspended rigidly at C.)

1. $AB = 10$, $BC = 24$, $AC = 31$, $W = 12$ tons.
2. $AB = 6$, $BC = 12$, $AC = 16$, $W = 6$ tons.
3. $AB = 15$, $BC = 50$, $AC = 45$, $W = 5$ tons.
4. $AB = 9$, $BC = 16$, $AC = 21$, $W = 4$ tons.

5. The jib of a crane is subjected to a compressive force equal to the weight of 24 tons, the suspended load being 10 tons. If the inclination of the jib to the horizontal is 60°, find the tension in the tie-rod. *Ans.* 16.1 tons.

6. In a crane the pull in the tie-rod inclined at an angle of 60° to the vertical is 18 tons. If the weight lifted be 8 tons, find the thrust in the jib. *Ans.* 23.06 tons.

7. In exercises 1–4, find the forces acting in each member of the crane when the load is suspended, but not rigidly, at the jib head, for

the two cases when the rope passes from the pulleys to the drum (*a*) parallel to the tie-rod, (*b*) parallel to the jib.

8. The jib of a crane is subjected to a compressive force equal to the weight of 4000 lbs., the suspended load being 2000 lbs. If the inclination of the jib to the horizontal is 45°, find the tension in the tie-rod.

9. In a crane the pull in the tie-rod inclined 45° to the vertical is 1000 lbs. Find the thrust in the jib if the weight is 2000 lbs.

10. In Ex. 9 find the thrust in the jib if the weight is 1000 lbs.

11. The thrust in the jib inclined 60° to the vertical is 1800 lbs. The load is 900 lbs. Find the tension in the tie-rod.

132. Polygon of Forces. The resultant of three or more concurrent forces lying in the same plane may be found by repeated applications of the triangle of forces.

Let a particle at O be acted upon by any number of forces, $F_1, F_2, \cdots$; to be definite, say F_1, F_2, F_3, F_4. To find their resultant proceed as follows. For the forces F_1 and F_2 construct the triangle of forces OAB (Fig. 81). Then OB is the

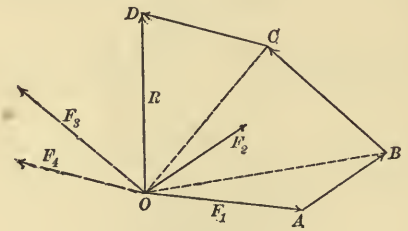


FIG. 81

resultant of F_1 and F_2. For the forces OB and F_3 construct the triangle of forces OBC. The sum is given by OC. In a similar manner combine OC and F_4. The resultant is $R = OD$.

The construction of the lines OB and OC is unnecessary and should be omitted. The figure $OABCDO$ is called the *polygon of forces*. OD, the closing side, is called the resultant. It will be noticed that the arrows on the vectors representing the

given forces all run in the same sense around the polygon, while the arrow of the resultant runs in the opposite sense.

If any number of forces acting at a point can be represented by the sides of a closed polygon taken in order, the point is in equilibrium and the resultant is zero.

From the above discussion we obtain the following rule for finding the resultant of any number of forces.

From any point O draw a line OA to represent in magnitude and direction the force F_1. From the extremity A draw a line AB to represent in magnitude and direction the force F_2. Continue this process for each of the given system of forces. Then the line which it is necessary to draw from O to close the polygon represents the resultant in magnitude and direction.

133. Resultant of Several Concurrent Forces. Analytic Formula. Let any number of forces F_1, F_2, $\cdots$, lying in the same plane, act on a particle at O. To fix the ideas, suppose there are three forces. With O as origin refer the forces to a pair of coördinate axes, OX and OY (Fig. 82). Resolve each

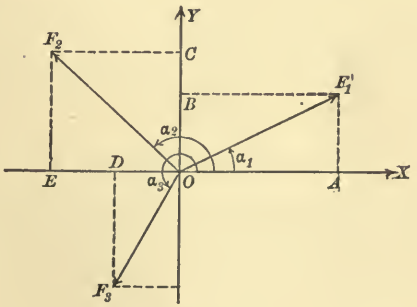


Fig. 82

force into two components, one along OX and one along OY. The components of F_1 will be OA and OB; of F_2, OC and OE;

of F_3, OD and OF. If α_1, α_2, α_3 represents the angles which F_1, F_2, F_3 make respectively with the axis OX, we have:

$$X_1 = OA = F_1 \cos \alpha_1, \qquad Y_1 = OB = F_1 \sin \alpha_1,$$
$$X_2 = OE = F_2 \cos \alpha_2, \qquad Y_2 = OC = F_2 \sin \alpha_2,$$
$$X_3 = OD = F_3 \cos \alpha_3, \qquad Y_3 = OF = F_3 \sin \alpha_3.$$

If a component acts upward or toward the right we will assume it to be positive; if downward or toward the left, negative.

The given system of forces is equivalent to another set consisting of the rectangular components of the forces of the given system. Let us use the letters X and Y to represent the sum of these components along the x-axis and the y-axis, respectively. Then

$$(7) \quad \begin{cases} X = F_1 \cos \alpha_1 + F_2 \cos \alpha_2 + F_3 \cos \alpha_3 \\ \quad = \text{the sum of all the horizontal components.} \\ Y = F_1 \sin \alpha_1 + F_2 \sin \alpha_2 + F_3 \sin \alpha_3 \\ \quad = \text{the sum of all the vertical components} \end{cases}$$

The two forces X and Y acting at right angles to each other are equivalent to the given system of forces. The single force R which is the resultant of X and Y is also the resultant of the given system of forces. We have

$$(8) \qquad R = \sqrt{X^2 + Y^2}.$$

The resultant R is always thought of as being positive. We now have the *magnitude* of the resultant force. To find the *line of action* we have

$$(9) \qquad \tan \alpha = \frac{Y}{X},$$

where α is the angle between the positive direction of the x-axis and the positive direction of the resultant R.

To find the *direction* along the line of action the two following equations are used:

$$\sin\alpha = \frac{Y}{R}, \qquad \cos\alpha = \frac{X}{R}. \tag{10}$$

It is obvious that equations (10) determine both the line of action and the direction along that line.

EXERCISES

1. If four forces of 5, 6, 8, and 11 units make angles of 30°, 120°, 225°, and 300° respectively, with a fixed horizontal line, find the magnitude and the direction of the resultant. *Ans.* 7.39; − 81° 6′.

2. Forces P, $2P$, $3P$, and $4P$ act along the sides of a square taken in order. Find the magnitude, the direction, and the line of action of the resultant.

Ans. $2\sqrt{2}P$, − 45° with line of force of $4P$, through $(-2a, -4a)$ where side of square is $4a$ and origin of coördinates is intersection of $3P$, $4P$.

3. A particle is acted on by five coplanar forces; a force of 5 lbs. acting horizontally to the right, and forces of 1, 2, 3, 4 lbs. making angles of 45°, 60°, 225°, and 300° respectively with the 5-lb. force. Find the magnitude and the direction of the resultant.

Ans. $R = 7.31$, $\theta = 334° 28'$.

4. Find the resultant of the following concurrent, coplanar forces:
(*a*) (14, 45°), (6, 120°), (5, 240°).
(*b*) (2, 0°), (3, 50°), (4, 150°), (5, 240°).
(*c*) (2, − 30°), (3, 90°), (4, 135°), (5, 225°).
(*d*) (5, − 30°), (6, 270°), (4, 120°), (3, 135°).

134. Resultant of Parallel Forces. Let F_1 and F_2 be two parallel forces acting in the same direction and with their points of application at the points A and B, Fig. 83. At A and B apply two equal and opposite forces, AS and BT, whose line of action coincides with AB. These will balance and will not change the effect of the other forces. Find the resultant AD of AS and F_1, and the resultant BE of BT and F_2, by constructing the parallelograms of forces. Then by constructing a

parallelogram of forces at O, the intersection of AD and BE produced, we may find their resultant OR, which is evidently the resultant of F_1 and F_2. Draw MK parallel to AB. Then

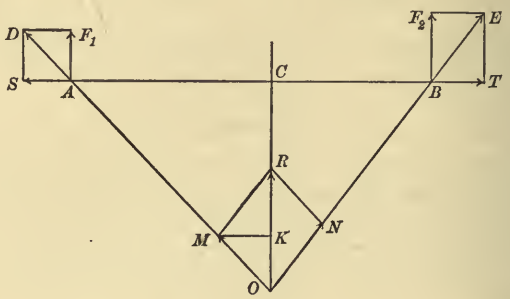


FIG. 83

since OM is equal to AD in magnitude and in direction and MR is equal to BE in magnitude and direction, it follows that the triangles OMK and ADF_1 are equal, and the triangles MKR and BTE are equal. Hence the resultant OR is equal to $F_1 + F_2$, and its line of action is a line through the point O parallel to the lines of action of F_1 and F_2.

Let C be the intersection of AB and OR. Then from the pairs of similar triangles OCA and AF_1D, and OCB and BF_2E, we have

$$\frac{AC}{OC} = \frac{AS}{F_1} \quad \text{and} \quad \frac{BC}{OC} = \frac{BT}{F_2} = \frac{AS}{F_2}.$$

Hence

$$\frac{F_1}{F_2} = \frac{BC}{AC}. \tag{11}$$

A similar proof can be given for the case of *unequal* parallel forces acting in *opposite* directions. Both results may be combined into the following theorem.

The resultant of any two parallel forces, acting in the same direction, or of two unequal forces acting in opposite directions,

is parallel to the forces and equal to their algebraic sum and cuts a line joining their points of application into segments, the lengths of which are inversely proportional to the magnitudes of the forces.

135. Moment of a Force. The ***moment of a force*** *with respect to a point,* called the ***center of moments,*** is the product of the magnitude of the force and the perpendicular distance, called the ***arm,*** from the point to the line of action of the force.

Geometrically the moment of a force is represented by twice the area of a triangle whose base is the line representing the given force and whose vertex is the center of moments.

The ***moment of a force*** in a given plane *with respect to a line* perpendicular to that plane is the moment of the force with respect to the foot of that perpendicular. The line is called the ***axis of moments.***

Moments are positive or negative according as they tend to produce counter clockwise or clockwise rotation about the axis of moments.

136. Composition of Moments. *The algebraic sum of the moments of any two forces with respect to any point of their plane is equal to the moment of their resultant with respect to the same point.*

There are two cases.

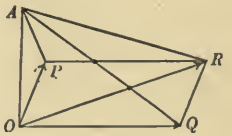


FIG. 84

CASE 1. *When the lines of action of the forces are not parallel.*

PROOF. Let OP, OQ be two forces acting at O, and OR their resultant; and let A be any point in the plane about which moments are to be taken. Join AO, AP, AQ, and AR. Then

$$\text{Area } \Delta AOQ = \text{Area } \Delta APR + \text{Area } \Delta RPO,^*$$

*By convention areas are positive or negative according as their boundaries are traversed in counterclockwise or clockwise direction.

since they have equal bases OQ and PR, and the altitude of ΔAOQ is equal to the sum of the altitudes of APR and RPO.

Area ΔAOR = Area ΔAOP + Area ΔAPR + Area ΔRPO, for obvious reasons it follows that

$$\text{Area } \Delta AOR = \text{Area } \Delta AOP + \text{Area } \Delta AOQ.$$

Therefore the moment of OR about A is equal to the sum of the moments of OP and OQ about A.

Frequently it is easier to determine the moment of a force by computing the sum of the moments of its components than to determine it directly.

Case II. *When the lines of action of the forces are parallel.* We exclude the case in which the forces are equal and opposite.

Suppose that two forces P and Q act on the body at the points A and B, Fig. 85. From any point O, draw $OACB$ perpendicular to the lines of action of the forces. Let $OA = p$, $AC = x$. Then by § 134, $CB = Px/Q$. Taking moments about O we find

$$\text{moment of } P = P \cdot p, \qquad \text{moment of } Q = Q(p + x + Px/Q),$$

$$\begin{aligned}\text{moment of } P + \text{moment of } Q &= Pp + Qp + Qx + Px \\ &= (P + Q)(p + x) = R(p + x) \\ &= \text{moment of } R.\end{aligned}$$

If P and Q are in opposite directions the proof is similar to the above and is left to the student. The proof in case P and Q are equal but opposite in direction is given in the following section.

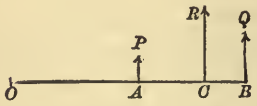


Fig. 85

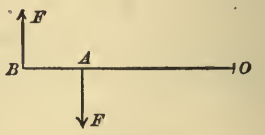


Fig. 86

137. Couples. A system of two parallel forces, equal in

magnitude and opposite in direction, is called a ***couple.*** The perpendicular distance between the lines of action of the forces is called the ***arm*** of the couple; and the plane containing the forces is called the plane of the couple.

The moment of a couple is the algebraic sum of the moments of its forces about any axis perpendicular to its plane and is equal to the product of either force and the length of the arm. For, let O be any axis, perpendicular to the plane of the couple, and OA and OB, the moment arms of the forces with respect to O. Taking moments about O, we have

$$F\cdot\overline{OB} - F\cdot\overline{OA} = F\cdot\overline{AB}.$$

The sign of the couple is plus if it tends to turn with clockwise rotation, and minus if it tends to turn with counter-clockwise rotation.

138. Conditions of Equilibrium.

(*a*) ***Concurrent coplanar forces.*** In order that the forces of a system may balance each other, the resultant must be equal to zero, that is

$$R = \sqrt{(\Sigma X)^2 + (\Sigma Y)^2} = 0. \tag{12}$$

Hence we have also

$$\Sigma X = 0, \quad \text{and} \quad \Sigma Y = 0. \tag{13}$$

The algebraic sum of the moments of the forces (written ΣM) about any point is equal to the moment of the resultant. If the forces are in equilibrium, $R = 0$; therefore

$$\Sigma M = 0. \tag{14}$$

These conditions are used in the second method of Ex. 1, below.

(*b*) ***System of parallel forces.*** If the algebraic sum of a system of parallel forces is not zero, the resultant is a single force and the system is not in equilibrium. Hence a necessary con-

dition for equilibrium is that

$$\Sigma F = 0,$$

where F represents the magnitude of a force. If the algebraic sum of the moments of the forces about any point is not zero, while the algebraic sum of the forces is zero, the resultant is a couple, and the body is not in equilibrium. Hence a necessary condition for equilibrium is that

$$\Sigma F \cdot x = 0, \tag{15}$$

where x is the moment arm of the force F.

EXERCISES

Balanced Systems of Forces acting through the same Point

1. A triangular frame ABC (Fig. 87) carries a load of 1000 lbs. at A. Find the stresses in the members AB and AC.

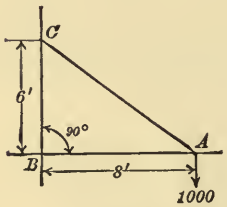


Fig. 87

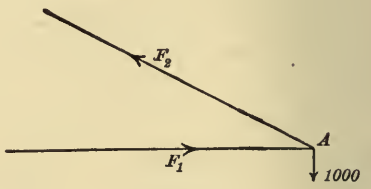


Fig. 88

Solution. We have in this problem a balanced system of forces acting through the point A, namely, the load of 1000 lbs. and the forces F_1 and F_2 in the members AC and AB. Both AC and AB are subjected to a compression. Hence both members exert a thrust in the direction indictated by the arrows. The problem is to determine the magnitude of two unknown forces in a balanced system of three forces, the directions of the forces being known. This problem may be solved in any one of the three following ways.

First Method. (*Triangle of Forces.*) The forces may be repre-

sented by the sides of a triangle taken in order, Fig. 88. If the figure is drawn to scale the magnitudes of the unknown forces F_1 and F_2 may be obtained directly from the figure by measurement.

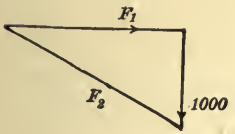


FIG. 89

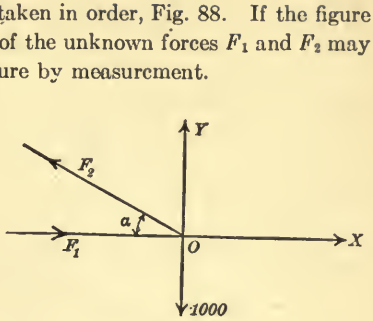


FIG. 90

If the lengths of all of the members of the frame ABC are known or can be computed, we can obtain the magnitudes of F_1 and F_2 by proportion, since the triangle ABC and the force triangle are similar.

In this particular problem we observe that the force triangle is right-angled and one acute angle is 60°. Hence

$$F_1 = 1000 \sin 60° = 866 \text{ lbs.}, \quad F_2 = 1000 \cos 60° = 500 \text{ lbs.}$$

SECOND METHOD. (*Resolution of Forces.*) Refer the forces to a system of coördinate axes, Fig. 88, and use the conditions (13) of equilibrium. We have

$$\Sigma X = F_2 \cos 30° - F_1 \cos 60° = 0,$$
$$\Sigma Y = F_2 \sin 30° + F_1 \sin 60° - 1000 = 0.$$

The solution of these equations gives,

$$F_1 = 866 \text{ lbs.}, \qquad F_2 = 500 \text{ lbs.}$$

THIRD METHOD. (*Moments.*) The sum of the moments of all the forces about any arbitrarily chosen point leads to one equation containing the unknowns. If we take the sum of the moments of all the forces about as many arbitrary points as there are unknowns then we will have as many equations as unknowns. The solution of these equations gives the magnitudes of the unknown forces. It is often advantageous to choose for the points about which moments are taken, points on the lines of action of the unknown forces, one on each line.

Taking moments about B we find

$$\Sigma M = 8\sqrt{3}F_1 - 1000 \times 12 = 0,$$

whence

$$F_1 = 866 \text{ lbs.}$$

Taking moments about C we find

$$\Sigma M = 1000 \times 4 - 8F_2 = 0,$$

whence

$$F_2 = 500 \text{ lbs.}$$

2. Find the stresses in the members AB and AC, of the triangular frame ABC, Fig. 91, the load at A being 1000 lbs.

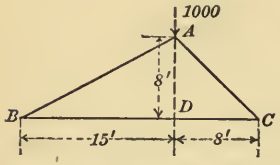


Fig. 91

[Hint. Use the triangle of forces.]

Ans. AB, 739.1 lbs.; AC, 922.2

3. Solve Ex. 2 (*a*) by using the method of resolution of forces; (*b*) by the method of moments.

4. Assuming that the frame in Ex. 2 is supported by a vertical force at B, find the magnitude of the force and the stress in BC.

5. A crane is loaded with 3000 lbs. at C. Determine the stresses in the boom CD, the tie BC, the mast BD and the stay AB, Fig. 92.

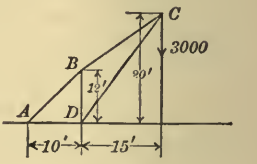


Fig. 92

[Hint. Use the triangle of forces.]

Ans. CD, 6250 lbs. (compression); BC, 4250 lbs. (tension); AB, 5858 lbs. (tension); BD, 2500 lbs. (compression).

6. Solve Ex. 5, using the method of resolution of forces.

7. Find the horizontal and vertical components of the supporting forces at A and D, Ex. 5.

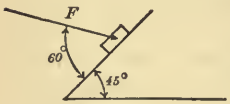


Fig. 93

8. Find the stresses in the members of the crane in Ex. 5, when the boom makes an angle of 15° with the horizontal.

9. What is the smallest force F which will prevent a weight of 150 lbs. from slipping down the incline represented in Fig. 93 if friction is neglected?

Ans. 212.2 lbs.

10. Let $F = 150$ lbs. (Fig. 93) and let the weight also be 150 lbs What will be the largest angle between the inclined plane and the horizontal at which the weight will not slip? *Ans.* 30°.

11. Experiments indicate that a horse exerts a pull on his traces equal to about one-tenth of his weight, when the working day does not exceed 10 hours. The draft of a certain wagon is due to (*a*) axle friction = 5 lbs. per 2000 lb. load; (*b*) gradient or hills; (*c*) rolling draft depending on height of wheel, width of tire, condition of road-bed, etc.

How large a load can a team of horses each weighing 1000 lbs. pull up a 10% grade if the rolling draft is zero. (A 10% grade is a rise of 10 feet for each 100 feet measured horizontally along the roadway.)

Ans. 1961 lbs.

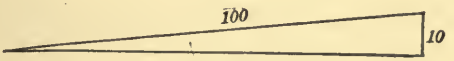


FIG. 94

12. What extra pull must a horse exert on his traces (assumed horizontal) if on a level road the wheel, 4 feet in diameter, strikes a stone 2 inches high, the load being 1000 lbs. *Ans.* 436 lbs.

13. A carriage wheel whose weight is W and whose radius is r rests on a level road. Show that any horizontal force acting through the center of the wheel greater than

$$P = W\frac{\sqrt{2rh - h^2}}{r - h}$$

will pull it over an obstacle whose height is h.

14. In Ex. 13, let $P = 100$ lbs., $W = 1000$ lbs., $r = 2$ feet. Find h. *Ans.* 0.126 in.

15. A 50 lb. boy swings on the middle of a clothes line which is 50 feet long. The lowest point is 2 feet below either end. Find the tension in the rope. *Ans.* 625 lbs.

16. A wire 90 feet long carries a weight of 25 lbs. at each of its trisection points. When the wire is taut each weight is 5 feet below the horizontal line connecting the points of support. Find the tension in each segment of the wire. *Ans.* 150; 147.9; 150 lbs.

17. Steam in the cylinder of an engine exerts a pressure of 20,000 pounds on the piston-head. The guides N, Fig. 95, are smooth. What

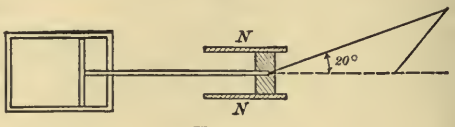
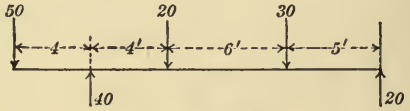


Fig. 95

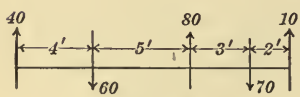
is the thrust in the connecting rod when it makes an angle of 20° with the horizontal? What is the pressure on the guides N? [Miller-Lilly]

Parallel Forces acting in the same Plane

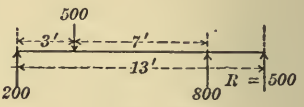
18. Determine the resultant R of each of the following systems of parallel forces.



(*a*) Fig. 96



(*b*) Fig. 97



(*c*) Fig. 98

19. Let AB (Fig. 99) represent a beam carrying the weights indicated and supported by the vertical forces F_1 and F_2. Find F_1 and F_2.

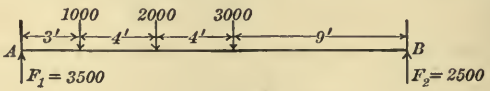


Fig. 99

20. The system of parallel forces in Fig. 100 is in equilibrium. Find the magnitudes and directions of the unknown forces F_1 and F_2.

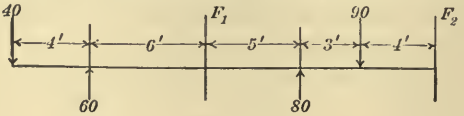


Fig. 100

21. If a horse exerts a pull on his traces equal to one-tenth of his weight, where should the single-tree for each of two horses weighing 1200 and 1600 lbs., respectively, be fastened to a double-tree in order that each horse shall do his proper share of the work?

22. The center clevis pin A, of a double-tree is a inches in front of the mid-point B, of the line connecting the end clevis pins C and D, which are b inches apart. If one horse is pulling c inches ahead of the other what fraction of the load L is each horse pulling, Fig. 101?

Ans. $\frac{1}{2}+\frac{ac}{b\sqrt{b^2-c^2}}$, $\frac{1}{2}-\frac{ac}{b\sqrt{b^2-c^2}}$.

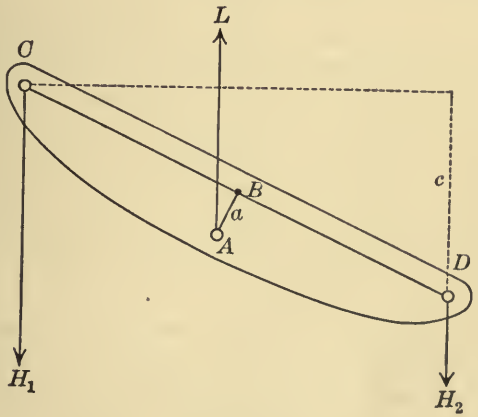


FIG. 101

23. Find what fractional part of the load each horse is pulling if $a = 2$, when

(*a*) $b = 41$, $c = 9$.
(*b*) $b = 39$, $c = 15$.
(*c*) $b = 34$, $c = 16$.
(*d*) $b = 52$, $c = 20$.
(*e*) $b = 37$, $c = 12$.
(*f*) $b = 50$, $c = 14$.
(*g*) $b = 61$, $c = 11$.
(*h*) $b = 36$, $c = 4$.

24. In Ex. 22, if the evener makes an angle θ with the tongue, what fractional part of the load is pulled by each horse?

Ans. $\frac{1}{2}+\frac{a}{b}\tan\theta$, $\frac{1}{2}-\frac{a}{b}\tan\theta$.

25. In Ex. 24 put $a = 2$, $b = 40$. Plot a curve using values of θ as abscissas and values of the load pulled by one horse as ordinates. What can you say about the part of the load pulled by this horse as θ increases?

26. In each of the cases of Ex. 23 find the pounds of pull exerted by each horse if the total pull on the load is 362.88 lbs.

27. The middle clevis pin A of a three-horse evener is a inches in front of the point B of the line connecting the end clevis pins C and D. The end clevis pins are b and $2b$ inches from the point B. Find what fractional part of the load is borne by the horse on the longer end when it is c inches behind the other horses.

$$Ans.\ \frac{1}{3}+\frac{1}{3}\frac{ac}{b\sqrt{9b^2-c^2}}.$$

28. Find what fractional part of the load the horse on the long end is pulling if $a = 2$, when

(*a*) $b = 24$, $c = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$.

(*b*) $b = 25$, $c = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$.

(*c*) $b = 26$, $c = 2, 4, 6, 8, 10, 12, 14$.

29. In Ex. 27, if the evener makes an angle θ with the tongue, what fractional part of the load is pulled by the horse on the long end.

$$Ans.\ \frac{1}{3}+\frac{1}{3}\frac{a}{b}\tan\theta.$$

30. In Ex. 29 put $a = 2$, $b = 25$. Plot a curve using values of θ as abscissas and fractional parts of the load pulled by the horse on the long end as ordinates. Discuss the problem.

31. A steel rail 60 ft. long weighs 1595 lbs. Where must a fulcrum be placed so that a 180 lb. man at one end can raise 4 tons at the other?

Ans. 6 ft.

CHAPTER VIII

SMALL ERRORS

139. Errors of Observation. Suppose that we measure the length of a building and record the result. Such a record is called a reading or an observation. Suppose that we measure the same length and record the reading on each of several successive days. On comparison it is likely we shall find that they do not exactly agree. What then is the true length? Whatever the actual length may be the difference between it and any observation of it is called an ***error of observation.***

Suppose that we measure the length of a building with a tape whose smallest division is one foot. If the length is not a whole number of feet, we estimate by the eye the fraction of a foot left over. This estimate will almost certainly be in error. If we measure the same length with a tape divided to eighths of an inch, the end of the building may coincide with a division of the tape or we may have to estimate the fraction of an eighth. Subsequent readings are not likely to agree exactly with the first, and even if they do all agree we cannot be sure that we have the true length. Inattention and lack of precision of the observer, inexperience in using the measuring instrument, or the use of an instrument which is defective or out of adjustment, all tend to introduce errors. It is important to keep in mind that such errors are always present, in greater or less degree, in every set of observations.

If a is the recorded reading of a measurement of an unknown quantity u, a measure of the error in this reading is a positive number m, such that u lies between $a - m$ and $a + m$. The actual error may be very much less than its measure m. For example if a rod of (unknown) length l be measured with a scale

divided to tenths of an inch and the reading is 47.8, it is fairly certain that $47.7 < l < 47.9$, and we write $l = 47.8 \pm 0.1$.

It is evident that any number will be in error if it is derived by computation from other numbers which are inexact. Approximations are used in computations not only for recorded measurements but also in the case of irrational numbers, such as surds, most logarithms, trigonometric functions, π, etc. We have 3-place, 5-place, 7-place, 10-place tables in order to secure the degree of accuracy desired in the computed result. In what follows it is shown how to find a measure of the error in a number computed by some of the simpler processes of arithmetic from given numbers the measures of whose errors are known.

140. **Error in a Sum.** Suppose that in measuring two quantities whose actual (and unknown) values are u and v, we make errors Δu and Δv respectively, and record the readings a and b. Then $u = a \pm \Delta u$, $v = b \pm \Delta v$ and their sum lies between $a + b - (\Delta u + \Delta v)$ and $a + b + (\Delta u + \Delta v)$.

Whence, $$u + v = a + b \pm (\Delta u + \Delta v).$$

That is, *the error in the sum of two readings is measured by the sum of their errors.*

This result is readily extended to the sum of more than two readings. The error in the difference of two readings is never greater than the sum of their errors, though it may be greater than their difference.

Example. Find the sum and difference of 46.8 ± 0.65 and 12.4 ± 0.15. Here the readings are 4.68, 12.4 and the measures of their errors are 0.65, 0.15 respectively. The measure of the error of their sum is

$$065 + 0.00 = 0.80;$$

whence $(46.8 \pm 0.65) + (12.4 \pm 0.15) = 59.2 \pm 0.8$

and $(46.8 \pm 0.65) - (12.4 \pm 0.15) = 34.4 \pm 0.8$

141. **Error in a Product.** With the same notation as above, the product uv lies between

$ab - (a\Delta v + b\Delta u + \Delta u \cdot \Delta v)$ and $ab + (a\Delta u + b\Delta u + \Delta u \cdot \Delta v)$,

whence, neglecting the small term $\Delta u \cdot \Delta v$, we have approximately,

$$uv = ab \pm (a\Delta v + b\Delta u).$$

That is, *a measure of the error in the product of two readings is the first times the error of the second plus the second times the error of the first.*

142. **Error in a Fraction.** The quotient of u divided by v evidently lies between

$$\frac{a - \Delta u}{b + \Delta v} \quad \text{and} \quad \frac{a + \Delta u}{b - \Delta v},$$

that is between

$$\frac{a}{b} - \frac{a\Delta v + b\Delta u}{b(b + \Delta v)} \quad \text{and} \quad \frac{a}{b} + \frac{a\Delta v + b\Delta u}{b(b - \Delta v)};$$

whence a measure of the error in the fraction is

$$\frac{a\Delta v + b\Delta u}{b(b - \Delta v)} = \frac{a\Delta v + b\Delta u}{b^2}, \text{ approximately.}$$

That is, *a measure of the error in the quotient of two readings is a measure of their product divided by the square of the divisor.*

Example. Find the product and quotient of 12.4 ± 0.15 and 46.8 ± 0.65. By § 141, a measure of the error in the product is $(12.4)(0.65) + (46.8)(0.15) = 15.08$ and the error in their quotient is measured by $15.08/(46.8)^2 = 0.0069$.

Whence, $(12.4 \pm 0.15)(46.8 \pm 0.65) = 580.32 \pm 15.08$

and $(12.4 \pm 0.15)/(46.8 \pm 0.65) = 0.265 \pm 0.0069$

EXERCISES

Make each of the following computations and state the result so as to show a measure of the error in it.

1. $(123 \pm 0.2) \pm (241 \pm 0.1)$.
2. $(222 \pm 0.5) \pm (111 \pm 0.4)$.
3. $(217 \pm 0.2)(117 \pm 0.3)$.
4. $(1267 \pm 0.5)(1342 \pm 0.4)$.
5. $(163 \pm 0.2)/(25 \pm 0.5)$.
6. $(732 \pm 0.3)/(21 \pm 0.4)$.
7. In Ex. 3 and 4 compute the term $\Delta u \cdot \Delta v$ neglected.
8. In Ex. 5 and 6 compute $\frac{a - \Delta u}{b + \Delta v}$ and $\frac{a + \Delta u}{b - \Delta v}$. Find the differ-

ence between the error thus computed and those computed in exercises 5 and 6 and consider the influence of this difference upon the quotient.

9. A line is measured with a chain (100 links each 1 ft. long). Afterwards, it is found that the chain is one foot too long. If the measured length was 10.36 chains, what is its true length if the error is assumed to be distributed through the chain? *Ans.* 10.4636 chains.

10. A line is measured with a 100-ft. tape and found to be 723.36 feet long. The tape is afterwards found to be 0.02 of a foot short. What is the true length of the line? *Ans.* 723.22 ft.

11. A certain steel tape is of standard length at 62° F. A tape will expand or contract sixty-five ten millionths of its length for each Fahrenheit degree change of temperature. A line is measured when the temperature of the tape is approximately 80° and found to be 323.56 feet long. What is its true length? Is it necessary to know the nominal or standard length of the tape to solve this problem? *Ans.* 323.52 ft.

12. What change in temperature is necessary to change a 100-foot tape by 0.01 of a foot, or 1 in 10,000? *Ans.* 15°.38

13. A certain 100-foot steel tape, standard length at 62° F., is used to measure from the monuments (Fig. 102) to the point A, in a line

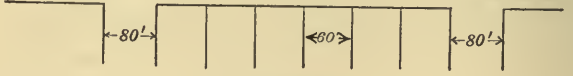


FIG. 102

between lots 2 and 3 extended, when the temperature is 40° F. Assuming that the map distances are correct, what lengths must be measured from 7th street and 8th street monuments respectively to locate the point A, the monuments being in the center lines of the streets? *Ans.* 160.02; 280.04

Show that if x, y, and z are small that

14. $(1 + x)(1 + y)$ is nearly equal to $1 + x + y$.

15. $(1 + x)/(1 + y)$ is nearly equal to $1 + x - y$.

16. $(1 + x)(1 + y)(1 + z)$ is nearly equal to $1 + x + y + z$.

17. Show that $(1 + 0.03)(1 - 0.05) = 0.98$ nearly.

18. Compute $(1.04)(1.06)(0.95)$. *Ans.* 1.05

19. Compute (*a*) $1.03/1.02$; (*b*) $(1.03)(1.02)$.

Ans. (*a*) 1.01; (*b*) 1.05

20. Compute (*a*) $(1.03)(0.98)$; (*b*) $1.03/0.98$.

Ans. (*a*) 1.01; (*b*) 1.05

21. Draw a figure (rectangle) to represent $(4.03)(9.02)$ and indicate 4×0.02; 9×0.03; 0.03×0.02; 4×9.

22. Show that the error in abc due to errors Δa, Δb, Δc in a, b, and c respectively, is $bc \cdot \Delta a + ac \cdot \Delta b + ab \cdot \Delta c$.

23. Compute $2.01 \times 4.02 \times 3.02$ Draw a figure (parallelopiped) to represent this product and indicate $3 \times 4 \times .01$; $2 \times 3 \times .02$; $2 \times 4 \times .02$; $.01 \times .02 \times .02$; $2 \times 4 \times 3$. *Ans.* 24.4

143. Data derived from Measurements. The preceding results apply immediately to the case in which numbers obtained by measurement are stated without any accompanying indication of the probable error.

In such cases it is understood that the given figures are all reliable, i. e., that we stop writing decimal places as soon as they are doubtful. The last figure written down should be as accurate as is possible. Then the error will surely not be more than 5 in the next place past the last one actually written.

Thus, if a certain length is reported to be 2.54 ft., we would understand that the true length is not more than 2.545 ft., and not less than 2.535 ft. For if the true length is more than 2.545 ft., it should be given as 2.55 ft.; and so on.

It may happen that the last figure written down is 0. This means that that place is reliable. Thus, to say that a given length is 2.4 ft. means that the true length is between 2.35 ft. and 2.45 ft. But to say that a given length is 2.40 ft. means that the true length is between 2.395 ft. and 2.405 ft.

In computations based upon numbers obtained by measurement, these facts must be kept in mind, and the result of any

calculation should not be stated to more decimal places than are known to be reliable.

Example 1. Find the area of a rectangle whose sides are found, by actual measurement, to be 2.54 ft. and 6.24 ft., respectively.

Since the error in writing 2.54 ft. may be as great as .005, we must write for the length of this side $(2.54 \pm .005)$ ft. Likewise, we must write for the other side $(6.24 \pm .005)$ ft. Hence, by the rule of § 141, the error in the product may be as large as

$$2.54 \times .005 + 6.24 \times .005,$$

that is .043. Hence we are not justified in expressing the answer to more than *one* decimal place; although

$$2.54 \times 6.24 = 15.8496,$$

we must sacrifice all the figures past 15.8, and write

$$2.54 \times 6.24 = 15.8 \pm .1$$

since the true answer may be as large as 15.894 Even the figure 8 in the first decimal place is not reliable, since the true area may be nearer 15.9 than 15.8 sq. ft.

EXERCISES

1. Assuming that the numbers stated below are the results of measurements, and that each of them is stated to the nearest figure in the last place, find the required answer and state it so that it also is correct to the nearest figure in the last place you give, or else to within a stated limit of possible error.

(*a*) $2.74 + 3.48 + 11.25 + 7.34$ *Ans.* 24.8

(*b*) $3.25 - 7.348 + 4.26 - 6.1$ *Ans.* $20.9 \pm .1$

(*c*) 6.27×3.14

(*d*) 26.5×11.4

(*e*) 7.32×5.4

(*f*) 36.4×2.78

(*g*) $61.54 \times 45.2 + 14.81$

(*h*) $8.26 \div 2.14$

(*i*) $43.7 \div 5.4$

(*j*) $(6.42 \times 2.35) \div 4.5$

2. The sides of a rectangle are measured, and are found to be 4 ft. 6.3 in. by 3 ft. 5.4 in. Express correctly the area of the rectangle.

3. The three sides of a rectangular block are measured and are found to be 7.4 in. by 3.6 in. by 4.7 in. Express the volume.

4. Suppose that the dimensions of a bin are measured roughly to the nearest foot, and that they are 8 ft. by 4 ft. by 3 ft. How large may the volume actually be? How small may it be?

Ans. 118.1 cu. ft., 65.6 cu. ft.

5. The floor of a room is found by measurement to be 22 ft. × 15 ft., each dimension being to the nearest foot. How should the area be stated? *Ans.* 330 ± 18 sq. ft., *or* 300 sq. ft.

6. If, in Ex. 5, the height of the room is 9 ft. to within the nearest foot, express the volume of the room.

144. Error in a Square. If a is an observed value of an unknown quantity u, then it follows directly from § 141 that a measure of the error in u^2 is approximately

$$a\Delta u + a\Delta u = 2a\Delta u, \text{ and we write}$$
$$u^2 = a^2 \pm 2a\Delta u.$$

That is, *a measure of the error in the square of a reading is twice the reading times its error.*

145. Error in a Square Root. With the same notation as above, $u = a \pm \Delta u$ is nearly equal to

$$a \pm \Delta u + \frac{\overline{\Delta u}^2}{4a},$$

since the last term is small. This is a perfect square and hence the positive square root of u is approximately

$$\sqrt{a} \pm \frac{\Delta u}{2\sqrt{a}}.$$

That is, *a measure of the error in the positive square root of a reading is equal to its error divided by twice its square root.*

EXAMPLE. Find $\sqrt{125 \pm 0.5}$ A measure of the error is

$0.5/2(11.18) = .022$ and $\sqrt{125 \pm 0.5} = 11.18 \pm 0.022$

Again $\sqrt{2400} = \sqrt{2401 - 1} = 49 - \frac{1}{98} = 49 - 0.0102$

146. Errors in Trigonometric Functions. Suppose a expressed in radians is an observed value of an unknown angle α.

Then $\alpha = a \pm \Delta\alpha$ and by § 94,

$$\sin \alpha = \sin (a \pm \Delta\alpha) = \sin a \cos \Delta\alpha \pm \cos a \sin \Delta\alpha.$$

Now if $\Delta\alpha$ is small, $\cos \Delta\alpha$ is nearly equal to 1, and $\sin \Delta\alpha$ is nearly equal to $\Delta\alpha$. Whence we have, approximately,

$$\sin \alpha = \sin a \pm \cos a \cdot \Delta\alpha,$$

and the smaller $\Delta\alpha$ is, the better the approximation. *Hence, a measure of the error in the sine of an angle is the error in the reading (expressed in radians) multiplied by the cosine of the reading.*

Similarly we can show that *a measure of the error in the cosine of an angle is the error in the reading multiplied by the sine of the reading.*

By means of these results and the principles of § 142 we can readily find a measure of the error in the other trigonometric functions. For example

$$\tan \alpha = \frac{\sin \alpha}{\cos \alpha} = \frac{\sin(a \pm \Delta\alpha)}{\cos(a \pm \Delta\alpha)}$$

and by § 142, a measure of the error in $\tan \alpha$ is

$$\Delta\alpha(\sin^2 a + \cos^2 a)/\cos^2 a = \sec^2 a \cdot \Delta\alpha$$

EXAMPLE. $\sin (36° 40' \pm 5') = \sin 36° 40' \pm .00145 \cos 36° 40'$
$= .5972 \pm .0012$

$\cos (36° 40' \pm 5') = \cos 36° 40' \pm .00145$ $\sin 36° 40' = .8021 \pm .0009$

$\tan (36° 40' \pm 5') = \tan 36° 40' \pm .00145$ $\sec^2 36° 40' = .7445 \pm .0023$

147. **Computation of Error from Tables.** This will be illustrated by an example. To find $\sin (36° 40' \pm 10')$ we look in a table of sines and find $\sin 36° 50' = .5995$, $\sin 36° 40' = .5972$, $\sin 36° 30' = .5948$; the difference between the first and second is .0023 and that between the second and third is .0024. Choosing the larger we write $\sin (36° 40' \pm 10') = .5972 \pm .0024$.

This method applies to tables of logarithms, squares, square roots, etc., in fact to any tables giving the values of a function

for regularly spaced values of the argument. For example, a measure of the error in $\log u = \log (a \pm \Delta u)$ is the greater of the differences $\log (a + \Delta u) - \log a$ and $\log a - \log (a - \Delta u)$. Thus to find $\log (17.4 \pm 0.7)$ we look up in the table $\log 16.7 = 1.2227$, $\log 17.4 = 1.2405$, $\log 18.1 = 1.2577$. The larger difference is 0.0178 and we write

$$\log (17.4 \pm 0.7) = 1.2405 \pm 0.0178$$

148. Errors in Computed Parts of Triangles. In practical applications, *e.g.* in surveying, the given parts of triangles are subject to errors of measurement and consequently the computed parts are also in error. Suppose the base AB of the triangle ABC in Fig. 103 is 23.4 ± 0.02, the side $AC = 15.6 \pm 0.04$, and the angle $A = 32° 30' \pm 10'$. Then the altitude

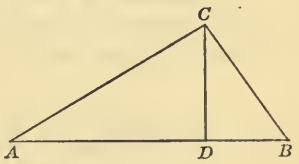


FIG. 103

$$\begin{aligned} CD &= (15.6 \pm 0.04) \sin (32° 30' \pm 10') \\ &= (15.6 \pm 0.04)(0.5373 \pm 0.0025) && \text{§§ 146, 147} \\ &= 8.382 \pm 0.060 && \text{§ 141} \end{aligned}$$

Again, the area is given by the following computation.

$$\text{Area} = \tfrac{1}{2}(23.4 \pm 0.02)(8.382 \pm 0.06) = 98.069 \pm 0.786.$$

Similarly a measure of the error in any computed part of a triangle may be found by the foregoing principles of this chapter.

EXERCISES

Calculate the error and the per cent. error of the square in each of the following numbers. Where no estimate of the error is expressed the error is supposed to be not greater than 5 in the next place past the last one written (§ 143).

1. $a = 76 \pm 0.1$
2. $a = 101 \pm 0.4$
3. $a = 32 \pm 0.04$
4. $a = 432 \pm 0.03$
5. $a = 2.46$
6. $a = 13.4$

Find the error and the per cent. error in the square root of each of the following:

7. 121 ± 0.4
8. 169 ± 0.5
9. 144 ± 0.02
10. 625 ± 0.01
11. 216 ± 0.03
12. 165 ± 0.2
13. 43.7
14. 6.45

15. Show that the error of the cube of $a \pm \Delta a$ is $\pm 3a^2 \cdot \Delta a$. Hence find a correct expression for the volume of a cube of side 2.6 ft.

16. Show that the error of the fourth power of $a \pm \Delta a$ is $\pm 4a^3 \cdot \Delta a$.

17. Show that the error of the cube root of $a \pm \Delta a$ is $\Delta a/3a^{2/3}$.

18. Find by the use of the tables and by use of the results of Ex. 17 the error in the cube root of (*a*) 1728 ± 2; (*b*) 15625 ± 1; (*c*) 343 ± 0.2

Ans. .005; .0006; .014

19. By applying twice the formula for the error of the square root of $a \pm \Delta a$, show that the error of the fourth root of $a \pm \Delta a$ is $\Delta a/4a$. Find the error in the fourth root of 256 ± 1. *Ans.* 0.001

20. Find the error by both methods of sin α for each of the following:

(*a*) $26° \pm 10'$.
(*b*) $45° \pm 15'$.
(*c*) $80° \pm 30'$.
(*d*) $10° \pm 10'$.

Ans. .0026; .0031; .0015; .0028

21. Find the error of (*a*) cos α; (*b*) tan α; (*c*) ctn α; (*d*) sec α; (*e*) csc α due to an error $\Delta\alpha$ in α.

22. Find by the use of the tables the error of (*a*) cos $(26° \pm 25')$; (*b*) tan $(20° \pm 3')$; (*c*) ctn $(70° \pm 20')$; (*d*) sec $(24° \pm 10')$; (*e*) csc $(46° \pm 10')$.

Ans. (*a*) .0032; (*b*) .0009; (*c*) .0066; (*d*) .0014; (*e*) .0039

23. Find the error of the area of the triangle for each of the following:

(*a*) $a = 120 \pm 0.3$ rod, $b = 144 \pm 0.2$ rod, $\gamma = 47° \pm 10'$.
(*b*) $a = 80 \pm 0.1$ rod, $b = 160 \pm 0.5$ rod, $\gamma = 89° \pm 30'$.
(*c*) $a = 40 \pm 0.5$ rod, $b = 60 \pm 0.3$ rod, $\gamma = 45° \pm 10'$.
(*d*) $a = 32 \pm 0.4$ rod, $b = 146 \pm 0.8$ rod, $\gamma = 26° \pm 5'$.

24. If A, B, C denote the angles and a, b, c the sides opposite in a plane triangle and if a, A, B are known by measurement, then

$$b = a \sin B / \sin A.$$

Show that the error, called the partial error in b due to a (written $\Delta_a b$), in the computed value of b due to an error Δa in measuring a is, approximately,

$$\Delta_a b = \sin B \cdot \csc A \cdot \Delta a.$$

Likewise show that

$\Delta_A b = -a \cdot \sin B \cdot \csc A \cdot \operatorname{ctn} A \cdot \Delta A$, and $\Delta_B b = a \cos B \cdot \csc A \cdot \Delta B$, and that the total error is, approximately,

$$\Delta b = \Delta_a b + \Delta_A b + \Delta_B b.$$

Note that A and B are to be expressed in radian measure.

25. The measured parts of a triangle and their probable errors are

$$a = 100 \pm 0.01 \text{ ft.}; \qquad A = 100° \pm 1'; \qquad B = 40° \pm 1'.$$

Show that the partial errors in the side b are

$$\Delta_a b = \pm 0.007 \text{ ft.}; \qquad \Delta_A b = \pm 0.003 \text{ ft.}; \qquad \Delta_B b = \pm 0.023 \text{ ft.}$$

If these should all combine with like signs, the maximum total error would be $\Delta b = \pm 0.033$ ft.

26. If $a = 100$ ft., $B = 40°$, $A = 80°$, and each is subject to an error of 1%, find the per cent. of error in b.

27. Find the partial and total errors in angle B, when

$$a = 100 \pm 0.01 \text{ ft.}, \qquad b = 159 \pm 0.01 \text{ ft.}, \qquad A = 30° \pm 10'.$$

28. The radius of the base and the altitude of a right circular cone being measured to 1%, what is the possible per cent. of error in the volume? *Ans.* 3%.

29. The formula for index of refraction is $m = \sin i / \sin r$, where i denotes the angle of incidence, and r the angle of refraction. If $i = 50°$ and $r = 40°$, each subject to an error of 1%, what is m, and what its actual and percentage error?

30. Water is flowing through a pipe of length L ft., and diameter D ft., under a head of H ft. The flow in cubic feet per minute, is

$$Q = 2356 \sqrt{\frac{HD^5}{L + 30D}}.$$

If $L = 1000$, $D = 2$, and $H = 100$, determine the change in Q due to an increase of 1% in H; in L; in D.

31. The formula for the area of a triangle in terms of its three sides

is $A = \sqrt{s(s-a)(s-b)(s-c)}$ where $s = \frac{1}{2}(a+b+c)$. A triangular field is measured with a chain that is afterwards found to be one link too long. The sides as measured are 6 chains, 4 chains, and 3 chains respectively. What is the computed area, and what is the true area?

32. Show that the erroneous area of a field, determined from measurements with an erroneous tape, will be to the true area as the square of the nominal length of the tape is to the square of its true length.

33. An irregular field is measured with a chain three links short. The area is found to be 36.472 acres. What is the true area?

34. The acceleration of gravity as determined by an Atwood's machine is given by the formula: $g = 2s/t^2$. Find approximately the error due to small errors in observing s and t.

Ans. $\Delta_s g = 2\Delta s/t^2$; $\Delta_t g = -4s/t^3$.

35. A right circular cylinder has an altitude 12 ft. and the radius of its base is 3 ft. Find the change in its volume (*a*) by increasing the altitude by 0.1 ft., and (*b*) the radius by 0.01 ft. (*c*) By increasing each simultaneously. *Ans.* (*a*) 2.83; (*b*) 3.02; (*c*) 5.85

36. The period of a simple pendulum is

$$T = 2\pi\sqrt{\frac{l}{g}}.$$

Show that $\Delta T/T = \frac{1}{2}\Delta l/l - \frac{1}{2}\Delta g/g$ and hence a small positive error of k per cent. in observing l will increase the computed time by $k/2\%$, and a small positive error of $k'\%$ in the value of g will decrease the computed time by $k'/2$ per cent.

37. Let w_1 denote the weight of a body in air, and w_2 its weight in water; then the formula

$$S = \frac{w_1}{w_1 - w_2}$$

gives the specific gravity of a body which sinks in water. If

$$w_1 = 16.5 \pm 0.01, \qquad w_2 = 12.3 \pm 0.02,$$

find the error in S due to the error in w_1; due to the error in w_2; the total error in S; the relative error $\Delta S/S$.

38. The specific gravity S of a floating body is given by the expression

$$S = \frac{w_1}{w_1 + (w_2 - w_3)},$$

where w_1 is the weight of the body in air, w_2 is the weight of a sinker in water, and w_3 is the weight in water of the body with sinker attached. Determine the specific gravity of a body and the probable error if

$$w_1 = 16.5 \pm 0.01$$
$$w_2 = 182.2 \pm 0.03$$
$$w_3 = 176.5 \pm 0.02$$ [Rietz and Crathorne]

39. To determine the contents of a silo I measure the inside diameter and height in feet and inches and find $D = 8$ ft. 2 in., $h = 21$ ft. 6 in. Find the error in the computed contents if there are errors $\Delta D = \pm 0.4$ in., $\Delta h = 0.3$ in. in the measured dimensions. *Ans.* 2.22 cu. ft.

40. My neighbor wants to buy the wheat from one of my bins. The measurements are: length = 12 feet; width = 6 feet; depth of wheat in bin = 8 ft. I make a mistake however of 1/4 inch in measuring each 2 feet of linear measure. Find the error of contents in cubic inches. Find the error in bushels if 2150.4 cu. in. make 1 bushel. A more accurate value is 2150.42 Find the error due to using 2150.4 instead of 2150.42 Find the error if 2150 is used.

41. I decide to sell to a neighbor by measurement my corn in the crib. I measure with a yard stick placing my thumb to mark the end of the yard and holding my thumb in place proceed to measure beyond it thus making an error of 1/2 inch. My measurements are length = 30 ft. 3 in.; width = 11 ft. 9 in.; height 13 ft. 6 in. Find the error in cubic inches due to my method of measuring.

42. The quantity of water in cubic feet per second flowing through a rectangular weir is given by the formula.

$$Q = 3.33[L - 2h]h^{3/2},$$

where h is the depth in feet of water over the sill of the weir, and L the length in feet of the sill. Find Q and the error in Q if $L = 26 \pm 0.1$, $h = 1.6 \pm 0.02$

CHAPTER IX

CONIC SECTIONS

149. Derivation. The ***circle***, the ***ellipse***, the ***parabola***, and the ***hyperbola***, are curves which can be cut out of a right circular conical surface by planes passing through it in various directions. For this reason, they are called also ***conic sections.*** Being plane curves, however, they can be defined and studied as the locus of a point moving in a plane under certain conditions.

150. The Circle. *The **circle** is the locus of a point moving at a fixed distance r from a fixed point C.*

The fixed distance r is called the ***radius;*** the fixed point C is called the ***center.***

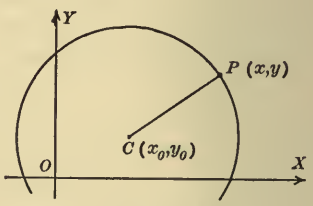


FIG. 104

EQUATION OF THE CIRCLE. Given the center, $C(x_0, y_0)$ and the radius, r, of a circle, to deduce its equation.

Let $P(x, y)$ be any point on the locus (Fig. 104). Then by (1) § 45,

$$\overline{CP} = \sqrt{(x - x_0)^2 + (y - y_0)^2},$$

and by the definition of the circle $\overline{CP} = r$. Hence, squaring and equating the two values of $\overline{CP}^2$, we find

$$(x - x_0)^2 + (y - y_0)^2 = r^2. \tag{1}$$

Conversely, let $Q(x_1, y_1)$ be any point which satisfies (1); i. e.,

$$(x_1 - x_0)^2 + (y_1 - y_0)^2 = r^2,$$

whence

$$\sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} = r,$$

but this says that $CQ = r$, and therefore Q is on the circle. Therefore (1) is the equation of the circle.

If the center is at the origin, $x_0 = y_0 = 0$, and the equation reduces to

(2) $$x^2 + y^2 = r^2.$$

151. Equation of the Second Degree. The most general equation of the second degree in x and y is of the form

(3) $$ax^2 + bxy + cy^2 + dx + cy + f = 0,$$

in which the coefficients are real numbers and a, b, c, are not all zero. The equation of the circle which we have obtained is of this form and has always $b = 0$ and $a = c$. Conversely, the special equation of the second degree

(4) $$ax^2 + ay^2 + dx + ey + f = 0.$$

is the equation of a circle or of no locus. To show this we have only to complete the square of the terms in x and of the terms in y. This process will reduce it to the form of (1) § 150, as is shown in the next paragraph.

152. Determination of Center and Radius. When the equation of a circle is given, the center and radius can be found by transposing the constant term to the right and completing the square of the terms in x and also of the terms in y.

Example 1. Find the center and radius of the circle

$$x^2 + y^2 - 3x - 2y - 3 = 0.$$

To reduce this equation to the form (1) we *complete the squares* as follows:

$$(x^2 - 3x + \ ) + (y^2 - 2y + \ ) = 3,$$
$$(x^2 - 3x + \tfrac{9}{4}) + (y^2 - 2y + 1) = 3 + \tfrac{9}{4} + 1,$$
$$(x - \tfrac{3}{2})^2 + (y - 1)^2 = (\tfrac{5}{2})^2$$

Comparing this with the standard equation (1), we see that the center is at (3/2, 1) and $r = 5/2$.

EXAMPLE 2. Examine the equation

$$9x^2 + 9y^2 - 6x + 12y + 6 = 0,$$

We complete the squares as follows:

$$x^2 + y^2 - \tfrac{2}{3}x + \tfrac{4}{3}y + \tfrac{6}{9} = 0,$$
$$x^2 - \tfrac{2}{3}x + \tfrac{1}{9} + y^2 + \tfrac{4}{3}y + \tfrac{4}{9} = -\tfrac{6}{9} + \tfrac{1}{9} + \tfrac{4}{9},$$
$$(x - \tfrac{1}{3})^2 + (y + \tfrac{2}{3})^2 = -\tfrac{1}{9}.$$

But since the square of a real number is positive (or zero), this shows that there are no points in the plane which satisfy the given equation. Therefore it has no locus.

EXAMPLE 3. Examine the equation

$$225x^2 + 225y^2 - 270x - 300y + 181 = 0.$$

We complete the squares as follows:

$$x^2 + y^2 - \tfrac{6}{5}x - \tfrac{4}{3}y + \tfrac{181}{225} = 0,$$
$$x^2 - \tfrac{6}{5}x + \tfrac{9}{25} + y^2 - \tfrac{4}{3}y + \tfrac{4}{9} = -\tfrac{181}{225} + \tfrac{9}{25} + \tfrac{4}{9},$$
$$(x - \tfrac{3}{5})^2 + (y - \tfrac{2}{3})^2 = 0.$$

This shows that the given equation is satisfied by the point (3/5, 2/3) and by no other point in the plane. This case may be looked upon as the limiting case of a circle whose center is at (3/5, 2/3), and whose radius is zero.

EXERCISES

1. Write the equation of the circle determined by each of the following conditions.

(*a*) Center (2, 4), radius = 3. (*b*) Center (− 1, 3), radius = 5.
(*c*) Center (−2, −3), radius = 3. (*d*) Center (3, − 2), diameter = 7.
(*e*) Center (*a*, *a*), diameter *a*. (*f*) Center (*r*, 0), radius = *r*.
(*g*) Center (4, 6) passes through the point (0, 3).
(*h*) Abscissa of center = 1, passes through the points (0, − 1), (0, 7).
(*i*) The segment from (1, − 3) to (7, 5) is a diameter.
(*j*) Center is on the line $x = y$, tangent to x-axis at (− 6, 0).

2. Write the equation of a circle of radius 6 when the origin is (*a*) at the highest point of the circle; (*b*) at the lowest point; (*c*) at the leftmost point; (*d*) at the rightmost point; (*e*) when the origin divides the horizontal diameter from left to right in the ratio 1/3.

3. Determine which of the following equations represent circles; find the center and the radius in each case.

(*a*) $x^2 + y^2 = 4x$. (*b*) $x^2 + y^2 = 6y$.
(*c*) $x^2 + 8y = 4x - y^2$. (*d*) $3x^2 + 3y^2 = 14y$.
(*e*) $x^2 + y^2 + 4x + 7 = 0$. (*f*) $x^2 + y^2 + 3x + 5y = 0$.
(*g*) $x^2 + y^2 = 2(y + 4)$. (*h*) $x^2 + y^2 = 4(x - 2)$.
(*i*) $x^2 + y^2 - 4x - 6y + 9 = 0$.
(*j*) $x^2 + y^2 + 101 = 87y - 20x$.
(*k*) $2x^2 + 2y^2 + 15y = 12x + 7$.
(*l*) $9x^2 + 9y^2 + 6y = 24x + 47$.
(*m*) $16x^2 + 16y^2 = 24x + 40y - 34$.
(*n*) $49x^2 + 49y^2 + 28x - 28y + 9 = 0$.
(*o*) $4a(ax^2 + bx - by) + b^2 + 4a(ay^2 - cx - cy) + c^2 = 0$.

4. Show that if the coefficients of x^2 and y^2 in the equation of a circle are each $+1$, the coördinates of the center can be found by taking *negative one-half the coefficient of x and negative one-half the coefficient of y.*

For example, the center of the circle

$$x^2 + y^2 - 5x + 4y - 3 = 0$$

is $(5/2, -2)$.

5. Find the coördinates of the center of each of the following circles, by the process of Ex. 4.

(*a*) $x^2 + y^2 - 4x - 6y + 9 = 0$. (*d*) $x^2 + y^2 - 2x + 4y + 1 = 0$.
(*b*) $x^2 + y^2 + 6x + 4y + 9 = 0$. (*e*) $x^2 + y^2 - 3x + 5y + 3 = 0$.
(*c*) $x^2 + y^2 - 4y = 0$. (*f*) $2x^2 + 2y^2 + 4x - 6y + 1 = 0$.

6. The value of the polynomial $P \equiv x^2 + y^2 - 2x - 4y + 3$ at any point of the xy-plane is found by substituting the coördinates of the point for x and y in P. Thus at (3, 2), $P = 2$. Show that all points at which P is positive lie outside a certain circle, and all points at which P is negative lie inside the same circle. With respect to this circle, where are the points (0, 1), (1, 2), (2, 3), (4, 5), (0, 3), (1, 4), (2, 2)?

153. Translation of Axes. Given a pair of axes OX and OY, a curve C, and its equation in terms of the coördinates $x = OA$ and $y = AP$. (Fig. 105.) Move the origin to the point O' whose coördinates referred to the old axes are (h, k) and draw new axes $O'X'$ and $O'Y'$ parallel to the old axes. The curve is not moved or changed but the coördinates of all its points are changed, and its equation is changed.

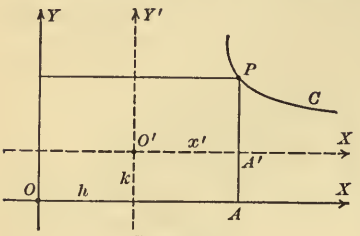


Fig. 105

From the figure we see that

$$x = x' + h$$

and

$$y = y' + k.$$

These equations are true no matter which way nor how far the origin is moved if the new axes are parallel to the old ones. These values substituted in the old equation of the curve, give the new equation. Hence, to find the new equation, substitute in the old equation, *in the place of x, the new x plus the abscissa of the new origin and in the place of y, the new y plus the ordinate of the new origin.*

Example. Translate the origin to the point $(1, -2)$ on the circle

$$3x^2 + 3y^2 - 5x + 2y = 6.$$

The new equation is

$$3(x' + 1)^2 + 3(y' - 2)^2 - 5(x' + 1) + 2(y' - 2) = 6,$$

and this reduces to

$$3x'^2 + 3y'^2 + x' - 10y' = 0.$$

154. Parabola. *The **parabola** is the locus of a point which moves so as to be always equidistant from a fixed point F and a fixed line L.*

The fixed point F is called the ***focus***. The fixed line L is called the ***directrix***.

155. Equation of the Parabola. Let F be the focus and RS the directrix of a parabola. (Fig. 106.) Draw FD perpendicular to the directrix. The midpoint O between D and F is on the parabola. Take O for the origin, OF for the x-axis, and take OY parallel to the directrix for y-axis. Let the distance $DO = OF = p$. Then the coördinates of the focus are (p, o). Let $P(x, y)$ be any point on the parabola. By definition, $FP = NP$; but

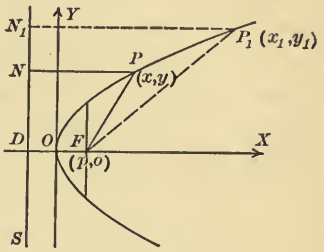


FIG. 106

$$FP = \sqrt{(x - p)^2 + y^2},$$

and

$$NP = x + p,$$

whence

$$\sqrt{(x - p)^2 + y^2} = x + p.$$

Squaring this, we find

(5) $$y^2 = 4px.$$

We have now proved that every point on the parabola satisfies the equation (5). It follows that the parabola has no points on the left of the y-axis, for negative values of x cannot satisfy the equation (5).

Conversely, let $P_1(x_1, y_1)$ be a point which satisfies (5); then

$$y_1^2 = 4px_1, \qquad \text{and} \qquad (x_1 - p)^2 = (x_1 - p)^2,$$

whence, adding, we have

$$(x_1 - p)^2 + y_1^2 = (x_1 + p)^2,$$

that is

$$\overline{FP_1}^2 = \overline{N_1P_1}^2.$$

Therefore P_1 is on the parabola. This completes the proof that (5) is the equation of the parabola.

The parabola is symmetric with respect to the line through its focus perpendicular to its directrix. This line is called the ***axis*** of the parabola. The point where the parabola crosses its axis is called its ***vertex.*** The chord through the focus perpendicular to the axis of the parabola is called its ***latus rectum.*** Let the student show that the length of the latus rectum is $4p$.

The parabola $y^2 = 4px$ crosses every horizontal line exactly once, and every vertical line to the right of the y-axis twice, once above and once below the x-axis. The farther the vertical line is to the right, the farther from the x-axis does the curve cut it.

By analogy to (5) it is evident that the equations of the parabolas shown in Figs. 107, 108, 109 are, respectively,

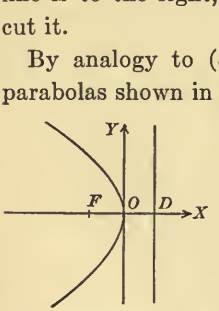


Fig. 107

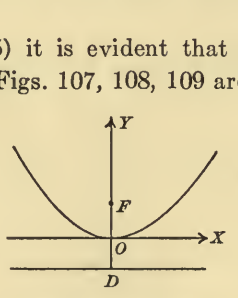


Fig. 108

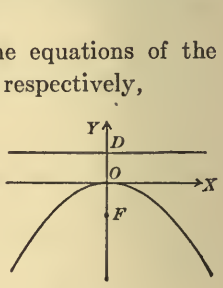


Fig. 109

(6) $y^2 = -4px$, (7) $x^2 = 4py$, (8) $x^2 = -4py$.

The position of each of these curves should be related to its equation as follows: *$y^2 = 4px$ is a parabola tangent to the y-axis at the origin, having its focus on the x-axis to the right.* The student should make similar statements concerning equations (6), (7), and (8).

156. Vertex not at the Origin. Each of the equations

$$(y - k)^2 = \pm 4p(x - h), \tag{9}$$

$$(x - h)^2 = \pm 4p(y - k) \tag{10}$$

represents *a parabola whose vertex is at* (h, k) *and whose axis is either horizontal* (equation (9)) *or vertical* (equation (10)). For, on translating the axes to this point they reduce to one of the types (5), (6), (7), or (8) considered above.

In particular, the equation

$$y = ax^2 + bx + c \qquad (a \neq 0) \tag{11}$$

represents a parabola whose axis is vertical. It is concave up or down according as a is positive or negative, and the vertex, focus, and directrix can be found by completing the square of the terms in x and reducing it to the form (10).

EXAMPLE 1. Locate the parabola $y = 2x^2 - 8x + 5$. Transposing,

$$2x^2 - 8x = y - 5;$$

dividing by 2,

$$x^2 - 4x = \tfrac{1}{2}y - \tfrac{5}{2};$$

adding 4,

$$x^2 - 4x + 4 = \tfrac{1}{2}y + \tfrac{3}{2};$$

or

$$(x - 2)^2 = \tfrac{1}{2}(y + 3).$$

Hence the vertex is the point $(2, -3)$, and $p = \frac{1}{8}$. The parabola is concave upwards; its focus is $\frac{1}{8}$ above the vertex, and its directrix is $\frac{1}{8}$ below the vertex.

EXAMPLE 2. Examine the equation $y = -2x^2 + 4x$. We may write successively the equations

$$x^2 - 2x = -\tfrac{1}{2}y, \qquad x^2 - 2x + 1 = -\tfrac{1}{2}y + 1, \qquad (x - 1)^2 = -\tfrac{1}{2}(y - 2).$$

Hence the vertex is at the point $(1, 2)$, and $p = \frac{1}{8}$. The parabola is concave downwards, its focus is $\frac{1}{8}$ below the vertex, and its directrix is $\frac{1}{8}$ above the vertex.

Similarly, the equation $x = ay^2 + by + c$ can be reduced to the type (9) by completing the square of the terms in y, and from this a sketch of the parabola can be made.

EXERCISES

1. Sketch each of the following parabolas, write the equation of its directrix, and the coördinates of its focus and vertex:

(*a*) $y^2 = 8x$. (*d*) $8y^2 = 3x$. (*g*) $(x + 3)^2 = 5(3 - y)$.
(*b*) $x^2 = 6y$. (*e*) $2y^2 = 25x$. (*h*) $x^2 = 10(y + 1)$.
(*c*) $y^2 = -3x$. (*f*) $(y - 2)^2 = 8(x - 5)$. (*i*) $(y + 4)^2 = -6x$.

2. Sketch each of the following parabolas, and find the coördinates of the vertex and focus and the equations of the directrix and axis.

(*a*) $y^2 - 2y - 4x + 6 = 0$. (*b*) $y^2 + 4y - 6x = 0$.
(*c*) $x^2 + 4x + 6y - 8 = 0$. (*d*) $x^2 - x + y = 0$.
(*e*) $4x^2 - 12x + 3y - 2 = 0$. (*f*) $3y^2 + 6y - 7x - 10 = 0$.

3. Sketch the parabolas with the following lines and points as directrices and foci, respectively; and find their equations.

(*a*) $x - 3 = 0$, $(6, -3)$. (*b*) $x = 0$, $(-2, -2)$.
(*c*) $y + 4 = 0$, $(-2, 0)$. (*d*) $y - 2b = 0$, $(0, 0)$.

4. Find the parabolas with the following points as vertices and foci, respectively.

(*a*) $(0, 0)$, $(2, 0)$. (*b*) $(1, 1)$, $(3, 1)$.
(*c*) $(-2, -2)$, $(-4, -2)$. (*d*) $(3, 2)$, $(3, 6)$.

5. Find the parabola with vertex at the origin and axis parallel to the x-axis, and passing through the point:

$(4, 1)$; $(2, 3)$; $(1, 1)$; $(-1, 2)$; $(2, -4)$; $(-2, -5)$.

6. The cable of a suspension bridge assumes the shape of a parabola if the weight of the suspended roadbed (together with that of the cables) is uniformly distributed horizontally. Suppose the towers of a bridge 240 ft. long are 60 ft. high and the lowest point of the cables is 20 ft. above the roadway. Find the vertical distances from the roadway to the cables at intervals of 20 ft.

7. An arch in the form of a parabolic curve is 29 ft. across the bottom and the highest point is 8 ft. above the horizontal. What is the length of a beam placed horizontally across it, 4 ft. from the top?

8. A parabolic reflector is 8 inches across and 8 inches deep. How far is the focus from the vertex? *Ans.* 2 in.

157. Ellipse. *An **ellipse** is the locus of a point which moves so that the sum of its distances from two fixed points is constant.*

The fixed points F and F' (Fig. 110) are called the ***foci.*** Let the constant distance be $2a$; this cannot be less than $F'F$. If it is just equal to $F'F$ the locus is evidently the segment $F'F$. Hence we assume that $2a > F'F$. Take the x-axis through the foci, and the origin midway between them. Then for all positions of the moving point P, we have

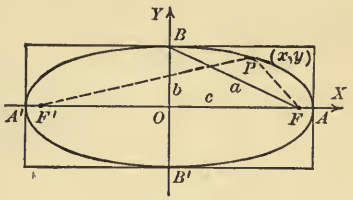


FIG. 110

$$F'P + FP = 2a. \tag{12}$$

One position of P is a certain point A on the x-axis to the right of F, and by (12),

$$F'A + FA = 2a$$

and

$$OA = \tfrac{1}{2}(F'A + FA) = a.$$

Similarly the point A' to the left of F' such that $A'O = a$, is a point on the ellipse. The points A and A' are called the ***vertices.*** The segment $A'A$ is called the ***major axis*** of the ellipse.

Another position of P is a point B on the y-axis above O and OB is denoted by b. By (12), we have

$$F'B + FB = 2a,$$

and since B is on the perpendicular bisector of $F'F$,

$$F'B = FB = a.$$

Similarly, the point B' below O such that $B'O = b$, is a point on the ellipse. The distance $B'B$ is called the ***minor axis.*** The

intersection of the major and minor axes is called the ***center*** of the ellipse.

The rectangle formed by drawing lines perpendicular to the major and minor axes at their extremities is called the ***rectangle on the axes.***

Let α denote the acute angle OFB. Then $\cos \alpha$ is called the ***eccentricity*** of the ellipse, and is denoted by e. It is evident that $e = OF/OA$. Hence, from the right triangle OFB, we have

$$0 < e < 1 \qquad \text{and} \qquad \frac{b^2}{a^2} = \sin^2 \alpha = 1 - e^2. \tag{13}$$

Since $OF = ae$ the coördinates of the foci F and F' are (ae, o) and $(-ae, o)$, respectively.

Then for all positions of the moving point P, by (12), we have

$$\sqrt{(x + ae)^2 + y^2} + \sqrt{(x - ae)^2 + y^2} = 2a. \tag{14}$$

Transposing the second radical, squaring, and reducing, we find

$$\sqrt{(x - ae)^2 + y^2} = FP = a - ex, \tag{15}$$

which is the right-hand focal radius.

Similarly, on transposing the first radical in (14), we obtain the equation

$$\sqrt{(x + ae)^2 + y^2} = F'P = a + ex, \tag{16}$$

which is the left-hand focal radius. Squaring either (15) or (16) and reducing, we find

$$(1 - e^2)x^2 + y^2 = a^2(1 - e^2), \tag{17}$$

whence, by (13),

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{18}$$

We have now proved that every point on the ellipse satisfies

(18). It can be proved, conversely, that every point which satisfies (18) is on the ellipse. Hence we may state the following theorem.

The equation of the ellipse whose semi-major axis is a, whose semi-minor axis is b, whose center is at the origin, and whose foci are on the x-axis, is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{19}$$

The numbers a, b, e, are positive, $a > b$, $e < 1$, $b^2/a^2 = 1 - e^2$. The coördinates of the foci are (ae, o) and $(-ae, o)$. The focal distances of any point on the ellipse are $a - ex$ and $a + ex$, respectively.

The equation shows that the curve is symmetric with respect to the x-axis and also with respect to the y-axis. It follows that the curve is symmetric with respect to the origin. It is only necessary to plot that part of the curve which lies in the first quadrant to determine the shape of the whole curve, which is as shown in Fig. 111.

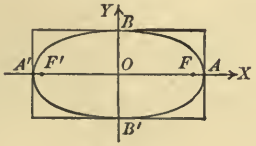


Fig. 111

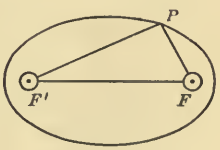


Fig. 112

The ellipse can be drawn by the continuous motion of a pencil point by means of a pair of tacks set at the foci and a loop of string around them as shown in Fig. 112. This is the best method of tracing an ellipse on a drawing board. It can be used to lay out an ellipse of any desired size on the ground. Let the student show that the length of the loop of string is $2a(1 + e)$.

158. Auxiliary Circle. A comparison of the equation of the ellipse (19) with that of the circle

$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1 \tag{20}$$

shows that *any ordinate of the ellipse is to the corresponding ordinate of the circle as b is to a.* The diameter of this circle (20) is the major axis of the ellipse. For this reason, the circle (20) is called the ***major auxiliary circle***, or simply the ***auxiliary circle.*** The points where any ordinate cuts the ellipse and the auxiliary circle are called corresponding points.

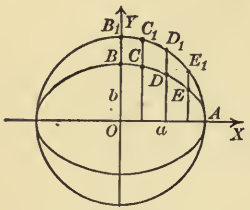


FIG. 113

159. Area of an Ellipse. Since the horizontal dimensions of the ellipse and its auxiliary circle are the same, and since their vertical dimensions are in the ratio $b : a$, we have

$$\frac{\text{Area of ellipse}}{\text{Area of auxiliary circle}} = \frac{b}{a}. \tag{21}$$

Hence, since the area of the circle is known to be πa^2, *the area of an ellipse whose semi-axes are a and b is* πab.

160. Projection. If a circle of radius a be drawn on a plane making an angle α with the horizontal plane, then the vertical projection of this circle on the horizontal plane is an ellipse whose semi-major axis is a and whose semi-minor axis is $a \cos \alpha$, since its ordinates are to the corresponding ordinates of the circle as $a \cos \alpha$ is to a.

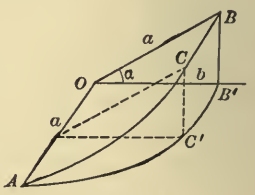


FIG. 114

EXAMPLE 1. Reduce the equation of the ellipse $3x^2 + 4y^2 = 48$ to standard form; find a, b, and c, the coördinates of the foci, the focal distances to the point (2, 3), and the area of the ellipse.

Dividing through by 48, we find

$$\frac{x^2}{16} + \frac{y^2}{12} = 1.$$

Then, by comparison with (19), we have $a^2 = 16$ and $b^2 = 12$, whence $a = 4$ and $b = 2\sqrt{3}$. From (13) we find $e = \frac{1}{2}$; hence $ae = 2$. It follows that the foci are $(-2, 0)$ and $(2, 0)$. The right-hand focal distance to (2, 3) is $a - ex = 3$ and the left-hand focal distance is $a + ex = 5$. The area is $\pi ab = 8\pi\sqrt{3} = 43.53$ +

EXAMPLE 2. Reduce the equation $15x^2 + 28y^2 = 12$.

Dividing by 12, we have

$$\frac{5x^2}{4} + \frac{7y^2}{3} = 1,$$

or

$$\frac{x^2}{(\frac{4}{5})} + \frac{y^2}{(\frac{3}{7})} = 1.$$

Hence, by comparison with (19), we have $a = \frac{2}{5}\sqrt{5}$ and $b = \frac{1}{7}\sqrt{21}$.

EXERCISES

1. Find the semi-axes, the eccentricity, locate the foci, and find the focal distances to any point (x, y) on the curve; construct the rectangle on the axes, and sketch the curve:

(*a*) $4x^2 + 9y^2 = 36$. (*b*) $x^2 + 25y^2 = 100$.
(*c*) $9x^2 + 25y^2 = 225$. (*d*) $9x^2 + 16y^2 = 144$.
(*e*) $x^2 + 2y^2 = 4$. (*f*) $6x^2 + 9y^2 = 20$.

2. In each of the following cases find the values of a, b, e, if they are not given. Locate the foci, and write the equation of the ellipse. Construct the rectangle on the axes and sketch the curve.

(*a*) $a = 10$, $b = 6$. (*g*) $b = 2\sqrt{3}$, $e = 1/2$.
(*b*) $a = 10$, $b = 8$. (*h*) $a = 5$, $e = 2/3$.
(*c*) $a = 5$, $b = 3$. (*i*) $a = 6$, $e = 0$.
(*d*) $a = 13$, $e = 12/13$. (*j*) $b = 8$, $e = 3/5$.
(*e*) $a = 7$, $e = 5/7$. (*k*) $b = 12$, $e = 5/13$.
(*f*) $a = 10$, $e = 3/5$. (*l*) $b = 2$, $e = 1/3$.

3. Find the area of each of the ellipses in Ex. 1.

4. Show that any oblique plane section of a circular cylinder is an ellipse.

5. Find the semi-axes and the area of the section formed by cutting off a log 14 inches in diameter by a plane making an angle of 60° with its length.

6. Design a flashing (sheet metal collar) for a four inch soil pipe projecting vertically through a roof whose pitch is 1/3.

7. A circular window in the south wall of a building is 4 ft. in diameter. Light from the sun passes through the window and falls on the floor. Find the area of the bright spot at noon, when the angle of elevation of the sun is (*a*) 60°, (*b*) 45°, (*c*) 30°.

8. An ellipse whose semi-axes are 10 and 9 is in a horizontal position. Through what angle must it be rotated about its minor axis in order that its projection on a horizontal plane shall be a circle.

Ans. 25° 50′.

161. Hyperbola. *A* ***hyperbola*** *is the locus of a point which moves so that the difference of its distances from two fixed points is constant.*

The fixed points are called the ***foci.*** Other terms are defined in a manner analogous to those for the ellipse.

By an analysis similar to that given in § 157 for the ellipse, it can be shown that the equation of the hyperbola whose semi-transverse axis is a, whose semi-conjugate axis is b, whose center is at the origin and whose foci are on the x-axis, is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \tag{22}$$

The curve consists of two branches and is symmetric with respect to both axes and with respect to the origin, as shown in Fig. 115. The quantities a, b, and e ($= \sec \alpha$), are positive, $a \lesseqgtr b$, $e > 1$, $b^2/a^2 = e^2 - 1$; the coördinates of the foci are (ae, o) and $(-ae, o)$; the focal distances to a point on the right branch are $ex - a$ and $ex + a$, and to a point on the left branch, the negatives of these. The diagonals OC and OC',

of the rectangle on the axes are called the ***asymptotes*** of the hyperbola, and the curve approaches nearer and nearer to

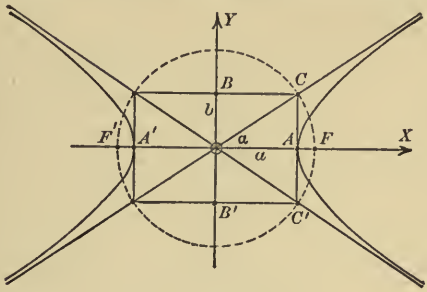


Fig. 115

them as the moving point recedes from the vertices. The equations of the asymptotes are

$$y = \frac{b}{a}x \qquad \text{and} \qquad y = -\frac{b}{a}x. \tag{23}$$

162. Rectangular or Equilateral Hyperbola. If the semi-axes of a hyperbola are equal, $b = a$, its equation reduces to the form

$$x^2 - y^2 = a^2. \tag{24}$$

The rectangle on the axes is a square, the eccentricity is sec 45° $= \sqrt{2}$, and the asymptotes are the two perpendicular lines $y = x$ and $y = -x$. This is called a ***rectangular*** or ***equilateral*** hyperbola. It plays a rôle among hyperbolas analogous to that played by the circle among ellipses.

The product of the distances of any point on an equilateral hyperbola to its asymptotes is constant. For the distance to the asymptote $y = x$ is $(x - y)\cos 45°$, and the distance to the asymptote $y = -x$ is $(x + y)\cos 45°$; hence the product of these distances is $a^2 \cos^2 45° = \frac{1}{2}a^2$.

It follows from this property that if the asymptotes of an equilateral hyperbola be taken for coördinate axes the equation of the curve will be

(25) $xy =$ **a positive constant,**

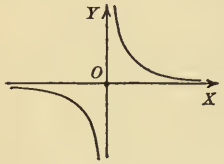


Fig. 116

when the branches are in the first and third quadrants, as shown in Fig. 116; and the equation will be

(26) $xy =$ **a negative constant,**

when the branches of the curve are in the second and fourth quadrants.

Example. Reduce the equation of the hyperbola $16x^2 - 9y^2 = 144$ to standard form.

Dividing by 144, we find

$$\frac{x^2}{9} - \frac{y^2}{16} = 1.$$

Hence, by comparison with (22), we have $a = 3$, $b = 4$. From $b^2/a^2 = e^2 - 1$ we find $e = 5/3$.

It follows that the coördinates of the foci are (5, 0) and (− 5, 0). The focal distances to a point on the right branch are

$$ex - a = \tfrac{1}{3}(5x - 9) \quad \text{and} \quad ex + a = \tfrac{1}{3}(5x + 9).$$

For example to the point $(6, 4\sqrt{3})$ they are 7 and 13. The equations of the asymptotes are

$$y = \tfrac{4}{3}x \quad \text{and} \quad y = -\tfrac{4}{3}x.$$

To sketch the curve, lay off $OA = 3$, $OB = 4$, Fig. 117, construct the rectangle on the axes, locate the foci by circumscribing a circle about this rectangle. Sketch in the curve free hand *in four parts* beginning each time at a vertex, using the asymptotes as guides, the curve approaching them in distance and direction.

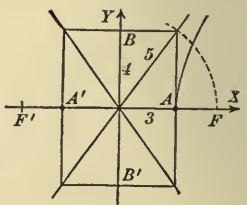


Fig. 117

EXERCISES

1. Find the semi-axes, the eccentricity, the coördinates of the foci, the focal distances to the point indicated, the equations of the asymptotes; construct the rectangle on the axes and the asymptotes, and sketch each of the following hyperbolas.

(*a*) $4x^2 - 9y^2 = 36$,	$(\sqrt{13}, 4/3)$.
(*b*) $4x^2 - y^2 = 8$,	$(-3/2, 1)$.
(*c*) $3x^2 - y^2 = 9$,	$(3, -3\sqrt{2})$.
(*d*) $3x^2 - 4y^2 = 1$,	$(-\sqrt{7}, \sqrt{5})$.
(*e*) $144x^2 - 25y^2 = 3600$,	$(10, -12\sqrt{3})$.
(*f*) $9x^2 - 16y^2 = 576$,	$(12, 3\sqrt{5})$.
(*g*) $25x^2 - y^2 = 100$,	$(-\sqrt{29}, 25)$.
(*h*) $225x^2 - 64y^2 = 14400$,	$(17, 28\frac{1}{8})$.
(*i*) $x^2 - y^2 = 9$,	$(-5, 4)$.
(*j*) $x^2 - y^2 = 400$,	$(101, 99)$.

2. Plot on the same axes the curves $xy = c$, for $c = 1, 4, 6, -1, -4, -6$.

3. Find the equation of the locus of a point which moves so that the difference of its distances from the two points $(1, 1)$ and $(-1, -1)$ is constant and equal to 2.

4. Find the locus as in Ex. 3, when the foci are (a, a) and $(-a, -a)$ and the constant is $2a$.

5. Find the locus of a point where two sounds emitted simultaneously at intervals one second apart at two points 2,000 ft. apart are heard at the same time, the speed of sound in air being 1,090 ft. per second.

6. On a level plain the crack of a rifle and the thud of the bullet on the target are heard at the same instant. The hearer must be on a certain curve; find its equation. (Take the origin midway between the marksman and the target.)

7. By translation of the axes (§ 153) find the equation of the ellipse

(*a*) whose foci are $(-4, 2)$ and $(0, 2)$, and whose eccentricity is $\frac{1}{2}$.

Ans. $3x^2 + 4y^2 + 12x - 16y = 20$.

(*b*) whose vertices are $(-2, 2)$ and $(4, 2)$, and which passes through the point $(1, 4)$. *Ans.* $4x^2 + 9y^2 - 8x - 36y + 4 = 0$.

(*c*) whose semi-axes are 5 and 3, whose right-hand focus is at $(4, -4)$, and whose left-hand vertex at $(-5, -4)$.

Ans. $9x^2 + 25y^2 + 200y + 175 = 0$.

[Hint. Start with the equation of the same curve when its center is at the origin.]

8. By the method of Ex. 7, find the equation of the hyperbola

(*a*) whose vertices are $(-2, 2)$ and $(4, 2)$, and whose eccentricity is $5/3$.

Ans. $16x^2 - 9y^2 - 32x + 36y = 164.$

(*b*) whose semiminor axis is 15, whose left-hand vertex is at $(-15, 3)$ and whose right-hand focus is at $(10, 3)$.

Ans. $225x^2 - 64y^2 + 3150x + 384y = 3951.$

(*c*) which passes through the origin and whose asymptotes are the lines $x = 2$ and $y = 1$. *Ans.* $xy = x + y.$

163. Intersection of Loci. If a point lies on a curve, its coördinates must satisfy the equation of that curve. Conversely, any pair of values of x and y which satisfy an equation determines a point on the locus of that equation. If the same pair of values of x and y satisfies two equations, it locates a point which is common to the two curves, i. e., a point of intersection. Hence, *to find the points of intersection of two curves, solve their equations simultaneously to find all their common solutions.*

Example 1. Find the intersections of the line $3x - y = 5$ and the ellipse $4x^2 + 9y^2 = 25$.

Solving the first equation for $y = 3x - 5$, substituting this in the second and reducing, we have

$$17x^2 - 54x + 40 = 0.$$

We can factor this quadratic by inspection:

$$(17x - 20)(x - 2) = 0,$$

whence

$$x_1 = 20/17 \quad \text{and} \quad x_2 = 2.$$

Substituting these values in the equation $3x - y = 5$, gives $y_1 = (-25/17)$ $y_2 = 1$. Therefore the points of intersection are $(20/17, -25/17)$, and $(2, 1)$.

Let the student plot the curves on the same axes and verify these results.

EXAMPLE 2. Where does the parabola

$$3y = x^2 - 5x + 12$$

intersect the ellipse

$$4x^2 + 3y^2 = 48?$$

Substituting the value of y from the first equation in the second and reducing, we get

$$x^4 - 10x^3 + 61x^2 - 120x = 0.$$

Factoring this equation, we have

$$x(x - 3)(x^2 - 7x + 40) = 0$$

and we see by inspection that $x_1 = 0$ and $x_2 = 3$ are roots. The quadratic $x^2 - 7x + 40$ has imaginary roots.

Substituting these values of x in the first given equation we find $y_1 = 4$ and $y_2 = 2$. Hence the points of intersection are $(0, 4)$ and $(3, 2)$.

EXERCISES

Find the points of intersection of the following pairs of curves.

1. $x^2 + 6xy + 9y^2 = 4$, $4x + 3y = 12$.
   Ans. $(14/3, -20/9)$, $(10/3, -4/9)$.
2. $x^2 - y^2 = 0$, $3x - 2y = 4$. *Ans.* $(4, 4)$, $(4/5, -4/5)$.
3. $y^2 + x = 0$, $2y + x = 0$. *Ans.* $(0, 0)$, $(-4, 2)$.
4. $x^2 - 5y = 0$, $x - y = 1$.
   Ans. $x = \frac{1}{2}(5 \pm \sqrt{5})$, $y = \frac{1}{2}(3 \pm \sqrt{5})$.
5. $2x + 3y = 5$, $4x^2 + 9y^2 + 16x - 18y - 11 = 0$.
   Ans. $(1, 1)$, $(-2, 3)$.
6. $x - y + 1 = 0$, $(x + 2)^2 - 4y = 0$. *Ans.* $(0, 1)$.
7. $y - 2x = 0$, $x^2 + y^2 - x + 3y = 0$.
   Ans. $(0, 0)$, $(-1, -2)$.
8. $x - 2y + 4 = 0$, $5x^2 - 4y^2 + 20 = 0$. *Ans.* $(1, 2\frac{1}{2})$.
9. $y = 2x - 3$, $4y^2 = (x + 3)(2x - 3)^2$.
   Ans. $(3/2, 0)$, $(1, -1)$.
10. $4y^2 = x^2(x + 1)$, $y^2 = x(x + 1)^2$. *Ans.* $(0, 0)$, $(-1, 0)$.
11. $2x^2 - 3y^2 = -58$, $3x^2 + y^2 = 111$.
    Ans. $(5, 6)$, $(-5, 6)$, $(5, -6)$, $(-5, -6)$
12. $x^2 = 4ay$, $y = 8a^3/(x^2 + 4a^2)$. *Ans.* $(\pm 2a, a)$.
13. $x^2 + y^2 = 2$, $x^2 + y^2 - 6x - 6y + 10 = 0$. *Ans.* $(1, 1)$.

164. Straight Line and Conic. The equations of the circle, parabola, ellipse, and hyperbola, are all of the second degree in x and y. Conversely, it can be shown that every such equation represents a conic section, if it represents any curve at all.

Given a straight line and a circle we know that one of three things will happen, 1) there may be two intersections, 2) there may be no intersection, or 3) there may be only one point in common and *then the line is a tangent.* The same three cases occur with the intersections of a straight line with any conic section.*

When we solve simultaneously the equation of a straight line with the equation of a conic, we may begin by substituting the value of y from the first equation in the second. The result is *a quadratic equation in* x. This quadratic equation (§§ 32, 33),

$$Ax^2 + Bx + C = 0 \tag{27}$$

will have 1) two real roots when $B^2 - 4AC > 0$, or 2) no real roots when $B^2 - 4AC < 0$, or 3) one real root when $B^2 - 4AC = 0$. These algebraic cases correspond exactly to the geometric cases enumerated above.

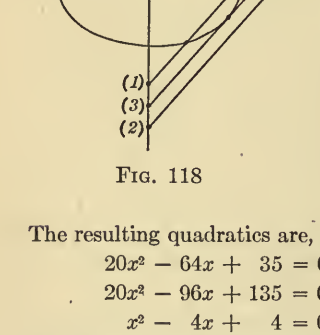


FIG. 118

EXAMPLE. Of the three parallel lines $8x - 9y = 20$, $8x - 9y = 30$, and $8x - 9y = 25$, the first cuts the ellipse $4x^2 + 9y^2 = 25$ in two points (5/2, 0) and (7/10, − 8/5), the second does not intersect it at all, and the third intersects it at (2, − 1) only, i. e. it is tangent at that point.

The resulting quadratics are, respectively,

$$20x^2 - 64x + 35 = 0, \qquad B^2 - 4AC = 1296,$$
$$20x^2 - 96x + 135 = 0, \qquad B^2 - 4AC = -1584,$$
$$x^2 - 4x + 4 = 0, \qquad B^2 - 4AC = 0.$$

*There is one exception to this rule: any line parallel to the axis of a parabola has one and only one point in common with the curve, but no such line is a tangent to the parabola.

EXERCISES

1. Show that one of the three lines $4x + 25 = 10y$, $4x + 27 = 10y$, $4x + 21 = 10y$, intersects the parabola $y^2 = 4x$ in two points, another is tangent, and the third does not intersect it at all.

2. Determine whether the following given lines are tangent, secant, or do not meet, the corresponding given conic.

(*a*) $x + y + 1 = 0$, $x^2 = 4y$.
(*b*) $x - 2y + 20 = 0$, $x^2 + y^2 = 16$.
(*c*) $2x + 3y = 8$, $y^2 = 4x$.
(*d*) $x + 2y = 5$, $x^2 + y^2 = x + 2y$.
(*e*) $2x = 3y$, $4x^2 - 3y^2 + 8x = 16$.
(*f*) $x + y = 8$, $4x^2 + y^2 = 16x$.

3. Find the points in which the circle $x^2 + y^2 = 45$ is cut by the lines (*a*) $2x - y = 15$, (*b*) $2x - y = 0$, (*c*) $2x - y = -15$.

Ans. (*a*) (6, −3), (*b*) (3, 6), (−3, −6), (*c*) (−6, 3).

4. Find the points in which the circle $x^2 + y^2 - 6x - 6y + 10 = 0$ is cut by the lines (*a*) $x + y = 2$, (*b*) $x + y = 6$, (*c*) $x + y = 10$, (*d*) $x - y = 0$.

Ans. (*a*) (1, 1), (*b*) (1, 5), (5, 1), (*c*) (5, 5), (*d*) (1, 1), (5, 5).

5. Find the points in which the parabola $3y = 2x^2 - 8x + 6$ is cut by the lines (*a*) $4x + 3y = 4$, (*b*) $4x + 3y = 6$, (*c*) $4x + 3y = 12$.

Ans. (*a*) (1, 0), (*b*) (0, 2), (2, $-\frac{2}{3}$), (*c*) (3, 0), (−1, $5\frac{1}{3}$).

6. Find the points in which the ellipse $3x^2 + 4y^2 = 48$ is cut by the lines (*a*) $x + 2y = 0$, (*b*) $x + 2y = 4$, (*c*) $x + 2y = 8$, (*d*) $x + 2y = -4$, (*e*) $x + 2y = -8$.

Ans. (*a*) $(2\sqrt{3}, -\sqrt{3})$, $(-2\sqrt{3}, \sqrt{3})$, (*b*) (4, 0), (−2, 3), (*c*) (2, 3), (*d*) (−4, 0), (2, −3), (*e*) (−2, −3).

165. Tangent and Normal. Focal Properties. The equation of the tangent to a conic can be found by the principles of § 164 if the slope of the tangent is known, or if the coördinates of one point on the tangent are known. This given point may be the point of contact or some other point through which the tangent is to pass.

The perpendicular to the tangent at the point of contact is called the ***normal*** to the conic at that point. When the slope

of the tangent is known or can be found, the equation of the normal can be written by the principles of §§ 59 and 61.

Example 1. Find the equation of the tangent to the parabola $y^2 = 24x$ which is perpendicular to the line $x + 3y + 1 = 0$.

By (13) § 61, the slope of the required tangent is 3, and by (11) § 59, $y = 3x + b$ is parallel to it no matter what value b has. Proceeding to find the points where this line intersects the parabola we are led to the quadratic equation,

$$9x^2 + 6(b - 4)x + b^2 = 0.$$

By § 32, this quadratic will have only one root and the line will be tangent to the parabola, if

$$36(b - 4)^2 - 36b^2 = 0.$$

This gives $b = 2$; whence, the equation of the required tangent is $y = 3x + 2$.

Example 2. Find the equation of the tangent and of the normal to the ellipse $3x^2 + 4y^2 = 48$ at the point (2, 3).

We first verify that the given point is in fact on the ellipse. Then by (10) § 59, $y - 3 = m(x - 2)$ is the equation of a line through (2, 3) no matter what value m has. Solving this simultaneously with the equation of the ellipse we get the quadratic equation,

$$(4m^2 + 3)x^2 + 8m(3 - 2m)x + 4(4m^2 - 12m - 3) = 0.$$

This equation will have only one root and the line will be tangent to the ellipse if (§ 32),

$$64m^2(3 - 2m)^2 - 16(4m^2 + 3)(4m^2 - 12m - 3) = 0,$$

that is if,

$$4m^2 + 4m + 1 = 0,$$

whence $m = -\frac{1}{2}$ and the equation of the required tangent is

$$y - 3 = -\tfrac{1}{2}(x - 2), \quad \text{or} \quad x + 2y = 8,$$

and the equation of the normal (whose slope by (13) § 61 is 2) is

$$y - 3 = 2(x - 2), \quad \text{or} \quad 2x - y = 1.$$

The normal at any point P on a parabola bisects the angle between the focal radius FP, and the line through P parallel to the axis of the curve.

We learn in Physics that light is reflected by a mirror in such a way that the *angle of incidence is equal to the angle of reflection.* Hence, a ray of light emanating from a source at the focus and

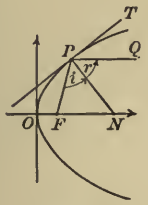


FIG. 119

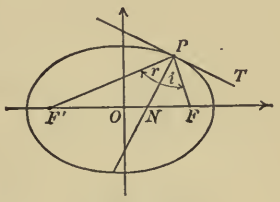


FIG. 120

striking the parabola at any point, will be reflected parallel to the axis. This is the principle of parabolic reflectors which are extensively used for head lights. It is easily seen that if the light be moved slightly beyond the focus, the reflected rays will tend to illuminate the axis.

The normal at any point of an ellipse bisects the angle between the focal radii to that point, Fig. 120. It follows that rays of light, or sound, emanating from one focus F, will after reflection by the ellipse, converge at the other focus F'. Hence the name *focus*. This is the principle of whispering galleries.

EXERCISES.

1. Find the equations of the tangents and normals to the following curves at the points indicated:

(*a*) $y^2 = 8x$, (2, 4), (*b*) $x^2 - y^2 = 64$, (10, 6),

(*c*) $x^2 + 3y^2 = 21$, (3, $-$ 2), (*d*) $28y^2 = 27x$, $(2\frac{1}{3}, 1\frac{1}{2})$.

2. Find the equations of the two tangents which can be drawn to the parabola $y^2 + 8x = 0$ from the point (2, 1) and verify that they are perpendicular.

3. Find the equations of the tangent and normal to the circle $x^2 + y^2 - 6x - 6y + 10 = 0$ at the point (1, 1).

Ans. $x + y = 2$, $x - y = 0$.

4. Find the equations of the tangents from the point (9, 3) to the circle $x^2 + y^2 = 45$. *Ans.* $2x - y = 15$, $x + 2y = 15$.

5. Find the equations of the tangent and normal to the circle $x^2 + y^2 = 6x + 2y$ at the point (2, 4).

Ans. $x - 3y + 10 = 0$, $3x + y = 10$.

6. Find the equations of the tangent and normal to the hyperbola $xy = 6$ at the point (2, 3). *Ans.* $3x + 2y = 12$, $2x - 3y + 5 = 0$.

7. Find the tangent to the parabola $y^2 = 12x$ which makes an angle of 60° with the x-axis.

8. Find the tangent to the parabola $y^2 = 6x$ which makes an angle of 45° with the x-axis.

9. Find the equations of the tangents to the circle

(*a*) $x^2 + y^2 = 4$ parallel to $2x + 3y + 1 = 0$,
(*b*) $x^2 + y^2 = 16$ parallel to $3x - 2y + 2 = 0$.

10. Find the tangents to the ellipse $9x^2 + 16y^2 = 144$ which make an angle of 30° with the x-axis.

11. Find the equations of the tangents to the following conics which satisfy the condition indicated.

(*a*) $y^2 = 4x$, slope $= 1/2$.
(*b*) $x^2 + y^2 = 16$, slope $= -4/3$.
(*c*) $9x^2 + 16y^2 = 144$, slope $= -1/4$.
(*d*) $x^2 = 4y$, passing through $(0, -1)$.
(*e*) $x^2 = 8y$, passing through $(0, 2)$.
(*f*) $x^2 + y^2 = 25$, at $(4, -3)$.
(*g*) $x^2 + 4y^2 = 8$, at $(-2, 1)$.
(*h*) $x^2 - y^2 = 16$, at $(-5, 3)$.
(*i*) $2y^2 - x^2 = 4$, at $(2, -2)$.
(*j*) $y^2 = 8x$, at $(2, 16)$.

12. Determine the condition for tangency of the following pairs of curves.

(*a*) $x^2 - y^2 = a^2$, $y = kx$. *Ans.* $k = \pm 1$.
(*b*) $x^2 + y^2 = r^2$, $4y - 3x = 4k$. *Ans.* $16k^2 = 25r^2$.
(*c*) $4x^2 + y^2 - 4x - 8 = 0$, $y = 2x + k$. *Ans.* $k^2 + 2k - 17 = 0$.
(*d*) $xy + x - 6 = 0$, $x = ky + 5$. *Ans.* $k^2 + 14k + 25 = 0$.

13. A parabolic reflector is 12 inches across and 8 inches deep. Where is the focus?

14. The ground plan of an auditorium is elliptic in shape. The extreme length is 2,725 ft. and the width is 2,180 ft. By what path will a sound made at one focus arrive first at the other focus, *i. e.*, directly or by reflection from the walls? How much sooner if sound travels 1,090 ft. per second?

166. Intersection of Conics. Simultaneous Quadratics. Two conics intersect, in general, in four points. Since their equations are of the second degree in x and y, this corresponds to the fact that two quadratics in x and y have, in general, four solutions. In some cases these solutions are not all real, or there may be less than four so that the conics represented intersect in less than four points.

As shown in Fig. 121*a*, the hyperbola $x^2 - y^2 = 5$ intersects the ellipse $x^2 + 4y^2 = 25$ in the four points (3, 2), (− 3, 2), (− 3, − 2), and (3, − 2). The parabola $4x^2 = 9y$ cuts the same ellipse only in (3, 2) and (− 3, 2), as shown in Fig. 121*b*.

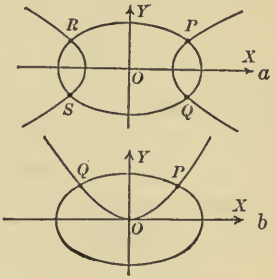


FIG. 121

Certain types of these equations can be solved by elementary methods. The most important cases will now be explained.

CASE I. *When all the terms (except the constant terms) are of the second degree in x and y.*

Eliminate the constant terms and factor the result into two linear factors.

EXAMPLE 1. $\begin{cases} x^2 - y^2 = 5, \\ x^2 + 4y^2 = 25. \end{cases}$

Multiplying the first equation by 5 and subtracting, we have

$$4x^2 - 9y^2 = 0,$$

whence

$$(2x - 3y)(2x + 3y) = 0.$$

Now solving simultaneously the two pairs of equations

$$(a) \begin{cases} x^2 - y^2 = 5, \\ 2x - 3y = 0. \end{cases} \qquad (b) \begin{cases} x^2 - y^2 = 5, \\ 2x + 3y = 0. \end{cases}$$

We find that the solutions of (*a*) are (3, 2) and (− 3, − 2); and those of (*b*) are (3, − 2) and (− 3, 2). It is easy to verify that these are all solutions of the given equations by actual substitution.

EXAMPLE 2. $\begin{cases} x^2 + 3xy = 28, \\ 4y^2 + xy = 8. \end{cases}$

Eliminating the absolute terms, we have

$$2x^2 - xy - 28y^2 = 0,$$

whence

$$(2x + 7y)(x - 4y) = 0.$$

This gives the two pairs of simultaneous equations

$$(a) \begin{cases} 4y^2 + xy = 8, \\ 2x + 7y = 0, \end{cases} \qquad (b) \begin{cases} 4y^2 + xy = 8, \\ x - 4y = 0. \end{cases}$$

The solutions are therefore $(14, -4)$, $(-14, 4)$, $(4, 1)$, $(-4, -1)$. Verify each of these by actual substitution.

SPECIAL METHOD, CASE I. If there is no term in xy the equations can be solved as linear equations considering x^2 and y^2 as the unknowns.

EXAMPLE. $\begin{cases} x^2 - y^2 = 5, \\ x^2 + 4y^2 = 25. \end{cases}$

Eliminate x^2 and solve for y^2. This gives $y^2 = 4$, whence $y = \pm 2$. Then eliminate y^2 and solve for x^2. This gives $x^2 = 9$, whence $x = \pm 3$. Verify that $(3, 2)$, $(-3, 2)$, $(-3, -2)$, $(3, -2)$, are all solutions of the given equations.

CASE II. When the equations are symmetric in x and y; i. e., when the interchange of x and y leaves the equations unchanged.

EXAMPLE. $\begin{cases} x^2 + y^2 = 13, \\ xy = 6. \end{cases}$

Substituting $s + t = x$ and $s - t = y$ in the given equations, we find

$$\begin{cases} 2s^2 + 2t^2 = 13, \\ s^2 - t^2 = 6. \end{cases}$$

Solving these equations, we have

$$s = \pm 5/2, \qquad t = \pm 1/2.$$

Hence the values of x and y are $x = \pm 3$ or ± 2, $y = \pm 3$, or ± 2. Testing these values in the given equations we verify that $(3, 2)$, $(2, 3)$ $(-2, -3)$, $(-3, -2)$, are solutions.

EXERCISES

Solve the following pairs of simultaneous equations.

1. $\begin{cases} 4x^2 + 4xy - y^2 = 7x - y, \\ 4x + 3y = 1. \end{cases}$

2. $\begin{cases} x^2 - y^2 + 16 = 0, \\ (x+1)^2 = (y+1)^2. \end{cases}$

3. $\begin{cases} 3x^2 + 4y^2 = 48, \\ y^2 = 3(1 - x). \end{cases}$

4. $\begin{cases} 5x^2 + 7y^2 = 225, \\ 2x + 3y = 9. \end{cases}$

5. $\begin{cases} 4x^2 + 3xy = 10, \\ 3y^2 + 4xy = 20. \end{cases}$

6. $\begin{cases} x^2 + y^2 = 153, \\ xy = 36. \end{cases}$

7. $\begin{cases} 2x^2 + 2xy + 5y^2 = 40, \\ x^2 - y^2 + 2x - 2y = 0. \end{cases}$

8. $\begin{cases} x^2 + y^2 - x - y = 204, \\ xy + x + y = 129. \end{cases}$

9. $\begin{cases} 2x^2 + xy + 3y^2 = 12, \\ 2x + y = 0. \end{cases}$

10. $\begin{cases} x^2 - 2y^2 + 1 = 0, \\ 2x^2 - 3y^2 - 23 = 0. \end{cases}$

11. $\begin{cases} 3x^2 + 4y^2 = 48, \\ x^2 + y^2 = 13. \end{cases}$

12. $\begin{cases} 4x^2 + 6xy + 4y^2 = 46, \\ x^2 + y^2 = 34. \end{cases}$

13. $\begin{cases} x^2 + 2xy = 407, \\ y^2 + 2xy = 455. \end{cases}$

14. $\begin{cases} x^2 + 2y^2 = 123, \\ y^2 + 2x^2 = 99. \end{cases}$

15. $\begin{cases} x^2 = 3x + 4y, \\ y^2 = 2x + 5y. \end{cases}$

16. $\begin{cases} 5x^2 + 3xy = 2, \\ 2x + y = 3. \end{cases}$

17. $\begin{cases} x(3x + y) = y(y + 3), \\ (3x - y)(x - 2y + 3) = 0. \end{cases}$

18. $\begin{cases} 3x(x - 4) = y - 5, \\ 2x + y = 30. \end{cases}$

19. $\begin{cases} x^2 - 5y^2 = 4, \\ x^2 + y^2 = 58 \end{cases}$

20. $\begin{cases} 3x^2 + 4y^2 = 48, \\ y + 3(x + 1) = 0. \end{cases}$

21. $\begin{cases} x^2 + xy + y^2 = 7, \\ x^2 - xy + y^2 = 19. \end{cases}$

22. $\begin{cases} 2x(2x - 3) = 184, \\ 9y(2x + y) = -135. \end{cases}$

23. $\begin{cases} x^2 - 3xy + y^2 = 1, \\ (x+y+2)(2x-y+1) = 0, \end{cases}$

24. $\begin{cases} x^2 + xy + y^2 = 7, \\ y^2 - x^2 = 5. \end{cases}$

25. $\begin{cases} x^2 + 3xy + y^2 - 4x - 2y - 1 = 0, \\ 15x + 4y - 1 = 0. \end{cases}$

26. $\begin{cases} x^2 + y^2 - x - y - 8 = 0, \\ x + 2xy + y - 17 = 0. \end{cases}$

CHAPTER X

VARIATION

167. Function and Variables. One of the most common scientific problems is to investigate the causes or effects of certain changes. The change or variation of one quantity in the problem is produced or caused by changes in other variable quantities and is said to depend upon, or be a function of these variables. Thus the growth of a plant depends on the amount of certain constituents in the soil, upon the temperature and humidity of the soil and of the atmosphere, upon the intensity of the light, and doubtless upon several other variables. The volume of gas contained in an elastic bag depends on the pressure and the temperature. The circumference of a circle depends only on the radius.

To study the effect of any one variable upon a function of two or more variables, we try to arrange conditions so that all the other variables of the problem shall remain constant, while this one varies. Thus we keep the temperature of a gas constant to find the effect on the volume of a change of the pressure. To study the effect of carbonate of lime on the growth of alfalfa, we arrange a series of plats of soil so that they shall have all the other constituents the same, and all be subject to the same conditions of light, heat, and moisture, but differ from plat to plat by known amounts of pulverized limestone.

The precise form of the relation between a function and its variables is often very complicated and difficult or impossible to obtain. Often, the best that can be done is to record the results of experiments and to study these records to deduce

general effects. Such results are called empirical. This is especially true of the so-called applications of science to the processes of nature.

168. Direct Variation. One of the simplest relations that can exist between two variables is called ***direct variation.*** When the *ratio* of two variables is constant, each is said to ***vary directly*** as the other.

The statement that *y varies directly as x* or simply *y varies as x*, is written

$$y \propto x,$$

which means that the ratio y/x is constant and implies the equation

$$y = kx,$$

where k is called the ***constant of variation.***

The circumference of a circle varies as the radius; i. e., $C \propto r$, or $C = kr$. The constant of variation is known to be $k = 2\pi = 6\frac{2}{7}$, approximately.

169. Inverse Variation. When the *product* of two variables is constant, each is said to vary inversely as the other. If y varies inversely as x, then

$$xy = k, \qquad \text{or} \qquad y = k\left(\frac{1}{x}\right),$$

whence y varies directly as $1/x$, the reciprocal of x.

EXAMPLE. The volume v, of a gas kept at constant temperature, varies inversely as the pressure p; i. e.

$$pv = k, \qquad \text{or} \qquad v = \frac{k}{p}, \qquad \text{or} \qquad p = \frac{k}{v}.$$

170. Joint Variation. When a function z depends upon two variables x and y, in such a manner that z varies as the product xy, i. e., $z = kxy$, then z is said to vary jointly as x and y. Thus, the area of a rectangle varies jointly as the length and the

breadth. This definition may be extended to functions of three or more variables. A function f, depending upon several variables $x, y, \cdots, z$, is said to vary jointly as x and $y, \cdots$, and z, when it varies as their product, i. e., $f = kx \cdot y \cdots z$. Thus, simple interest varies jointly as the principal, and the rate, and the time.

It is evident that if one variable z depends on two other variables x and y, and if z varies as x when y is constant, and z varies as y when x is constant, then z varies jointly as x and y when x and y vary simultaneously. Thus, the area of a triangle varies as the altitude when the base is constant and varies as the base when the altitude is constant; therefore the area varies jointly as the base and the altitude.

This principle is readily extended to functions of three or more variables. Thus, simple interest varies as the principal when rate and time are constant, as the rate when principal and time are constant, and as the time when principal and rate are constant; therefore simple interest varies jointly as the principal, the rate, and the time.

171. Graphic Representation. When y varies directly as x, the graph of the relation, $y = kx$, which connects them is

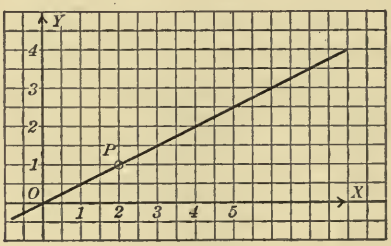


Fig. 122

a straight line through the origin whose slope is k. The position of this line is fixed and the value of k can be determined if we

know one other point on the line, i. e., one pair of simultaneous values of x and y; and values of y corresponding to any given values of x, can be read directly from the graph. Then k is the difference of two values of y divided by the difference of the corresponding values of x (§ 58).

EXAMPLE. Given that y varies as x and that $y = 1$ when $x = 2$.

Plotting the point (2, 1) and drawing the line OP we have the graph of the relation between x and y. From this we read off $y = \frac{1}{2}$ when $x = 1$, $y = 3\frac{1}{4}$ when $x = 6\frac{1}{2}$, etc. Fig. 122.

When y varies inversely as x, the graph of their relation $xy = k$ is *a rectangular hyperbola* asymptotic to the x and y axes. Here again one point is sufficient to determine k and fix the curve.

EXAMPLE. Given that volume v, varies inversely as pressure p, and that $v = 12$ when $p = 3$.

Then $pv = k$, $3 \cdot 12 = k$, $pv = 36$. The graph of this is shown in

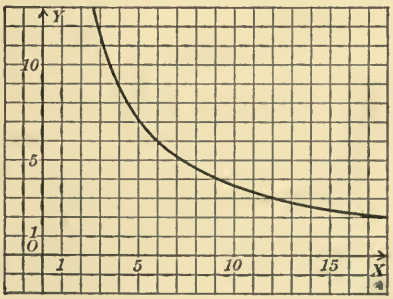


FIG. 123

Fig. 123 for positive values of p and v. From this we can read off $v = 6$ when $p = 6$, $v = 4$ when $p = 9$, etc.*

* When z varies jointly as x and y, the graph of the relation $z = kxy$, in three dimensions, is a *surface* called a hyperbolic paraboloid with which the student is not yet familiar.

172. Determination of the Constant. By substituting in an equation of variation a set of simultaneous values of the variables, the constant of variation can be determined.

EXAMPLE 1. Given, y varies as x and $y = 8$ when $x = 10$. We may write $y = kx$, as in § 168. Substituting $x = 10$ and $y = 8$, we find $8 = k \cdot 10$. From this equation, we can find k by dividing both sides by 10. This gives $k = 4/5$. Hence we have $y = (4/5)x$.

From this equation, the value of y corresponding to any given value of x can be found. Thus, $y = 1\frac{3}{5}$ when $x = 2$.

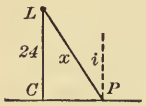


FIG. 124

EXAMPLE 2. A light is 24 inches above the center of a table. The illumination I at any point P of the surface of the table varies directly as the cosine of the angle of incidence, i, of the ray LP, and also inversely as the square of the distance $LP = x$ to the light. If the illumination at C is 10, what is it at any point P of a circle of radius 18 inches about C?

SOLUTION. The illumination I at any point is

$$I = k\frac{\cos i}{x^2},$$

but $x = 24 \sec i$ and therefore

$$I = \frac{k}{576}\cos^3 i.$$

Since $I = 10$ when $i = 0$, $k = 5760$, and hence

$$I = 10 \cos^3 i.$$

Now when $CP = 18$, $\cos i = 4/5$, and I at P is equal to 5.12

EXERCISES

1. Write equations equivalent to each of the following statements; determine the constant of variation and construct the graph.

(*a*) y varies as x; $y = 7$ when $x = 3$.
(*b*) y is proportional to x; $y = 3$ when $x = 2\frac{1}{2}$.
(*c*) y varies inversely as x; $y = 1\frac{1}{2}$ when $x = 1\frac{1}{3}$.
(*d*) v varies inversely as p; $v = 3$ when $p = 2$.

2. Write equations equivalent to each of the following statements and find the value asked for in each case.

(*a*) y varies as x^2; $y = 81$ when $x = 3$; find y when $x = 5\frac{1}{3}$.

(*b*) y varies as $\sin x$; $y = 2$ when $x = 30°$; find y when $x = 150°$.

(*c*) u varies inversely as v; $u = 8$ when $v = 2$; find u when $v = 6$.

(*d*) z varies jointly as x and y; $z = 6$ when $x = 2$, $y = 7$; find z when $x = 4$, $y = 6$.

(*e*) y varies directly as r and inversely as s; $y = 16$ when $r = 10$, $s = 8$; find y when $r = 7$, $s = 12$.

(*f*) u varies jointly as x, and y^2, and z^{-1}; $u = 6$ when $x = 2$, $y = 3$, $z = 4$; find u when $x = 10$, $y = 15$, $z = 25$.

(*g*) z varies directly as x and inversely as y^2; $z = 2$ when $x = 17$, $y = 3$; find x when $z = 6$, $y = 4$.

3. Express each of the following by means of an equation.

(*a*) The volume of a cone varies directly as the height when the radius of the base is constant.

(*b*) The volume of a cone varies directly as the square of the radius of the base when the height is constant.

(*c*) The number of calories of heat produced when a moving body is stopped varies jointly as the mass and the square of the velocity.

(*d*) The squares of the periods of the planets vary directly as the cubes of their mean distances from the sun.

4. With the statement of Ex. 3 (*c*) find the heat generated by a mass of 8 kilograms striking the sun with a velocity of 500 miles per second if a body weighing one kilogram and moving with a velocity of 380 miles per second on striking the sun produces 45,000,000 calories of heat.

5. The simple interest due on P dollars varies jointly as the amount P, the rate, and the time. If \$1000 yields \$30 interest in six months find the interest on \$1200 for eight months at 7%.

6. The amount of heat received by a given planet varies inversely as the square of its distance from the sun and directly as the square of its radius.

(*a*) What is the effect of doubling the distance?

(*b*) Mercury has a diameter of 3000 miles and is 36 million miles from the Sun. The Earth has a diameter of 8000 miles and is 93 million miles from the Sun. Compare the amounts of heat they receive.

7. With the statement in Ex. 3(*d*), taking the distance of the Earth from the Sun as the unit and the period of the Earth as the unit of time,

find the period of Neptune whose distance from the Sun is known to be 30 units. *Ans.* 165 yrs.

8. The amount of heat received on a surface of given size varies inversely as the distance from the source. One body is twice as far as another from the source. Compare the amounts of heat received.

9. The resistance offered to a rifle bullet varies directly as the square of the velocity. Discuss the effect of doubling the velocity.

10. The maximum load P that a rectangular beam supported at one end will hold without breaking varies directly as the breadth, the square of the depth and inversely as the length. A beam $4'' \times 2'' \times 10'$ supports 300 pounds. What load will the same beam support when placed on edge?

11. The deflection y in a rectangular beam supported at the ends and loaded in the middle varies directly as the cube of the length, inversely as the breadth, and inversely as the cube of the depth. A beam 6 inches wide, 8 inches deep, 15 feet long, supporting 1000 lbs., has a deflection of $\frac{1}{2}$ inch at the middle. Find the deflection in a beam 4 inches wide, 4 inches deep, 10 feet long, supporting 800 lbs.

12. With the data of Ex. 10, find the load which a beam 4 inches wide, 6 inches deep, and 16 feet long will support.

13. With the data of Ex. 10, find the longest beam 4 inches wide and 4 inches deep which will support 100 lbs.

14. With the data of Ex. 10, find the least depth of a beam 12 feet long and 4 inches wide that will support 400 lbs.

15. With the data of Ex. 10, find the least breadth of a beam 12 feet long and 4 inches deep that will support 500 lbs.

16. Evaporation from a surface varies directly as its area.

(*a*) Of two square vats the side of one is 10 times that of the other. What is the ratio of evaporation?

(*b*) Of two circular vats one evaporates 10 times as fast as the other. Compare their radii.

17. The distance traversed by a falling body varies directly as the square of the time. If a body falls 144 feet in 3 seconds, how far will it fall in 5 seconds?

18. The area of a triangle varies jointly as the length of the base b and the altitude a. Write the law if the area is 12 square inches when $a = 6$ inches and $b = 4$ inches.

19. Similar figures vary in area as the squares of their like dimensions. A new grindstone is 48 inches in diameter. How large is it in diameter when one-fourth of it is ground away?

20. A circular silo has a diameter of a feet. What must be the diameter of a circular silo of the same height to hold 4 times as much?

21. What is the effect on the area of a regular hexagon if the length of each side of the hexagon is doubled.

22. Similar solids vary in volume as the cubes of their like dimensions. A water pail that is 10 inches across the top holds 12 quarts. Find the volume of a similar pail that is 12 inches across the top.

23. Using the rectangular pack, 432 apples 2 inches in diameter can be put in a box $12 \times 12 \times 24$. How many 3 inch apples can be packed in the same box? How many 4 inch apples? *Ans.* 128; 54.

24. If a lever with a weight at each end is balanced on a fulcrum, the distances of the two weights from the fulcrum are inversely proportional to the weights. If 2 men of weights 160 lbs. and 190 lbs. respectively are balanced on the ends of a 10 foot stick, what is the length from the fulcrum to each end? *Ans.* $4\frac{5}{7}$ ft.; $5\frac{2}{7}$ ft.

25. A wire rope 1 inch in diameter will lift 10,000 lbs. What will one $\frac{3}{8}$ inches in diameter lift? *Ans.* 1,406 lbs.

26. Two persons of the same *build* are similar in shape; their weights should vary as the cube of their heights. A man $5\frac{1}{2}$ ft. tall weighs 150 lbs. Find the weight of a man of the same build and 6 feet tall.

Ans. 194.74 lbs.

27. A man 5 ft. 5 in. tall weighs 140 lbs., and one 6 ft. 2 in. tall weighs 216 lbs. Which is of the stouter build?

28. The size of a stone carried by a swiftly flowing stream varies as the 6th power of the speed of the water. If the speed of a stream is doubled, what effect does it have on its carrying power? What effect if trebled?

CHAPTER XI

EMPIRICAL EQUATIONS

173. Empirical Formulas. In practice, the relations between quantities are usually not known in advance, but are to be found, if possible, from pairs of numerical values of the quantities discovered from experiment.

In order to determine the relation between these quantities it is useful to first plot the corresponding pairs of values upon cross-section paper, and draw a smooth curve through the plotted points. If the curve so drawn resembles closely one of the following types of curves:

(1)	$y = ax + b$	(*straight line*),
(2)	$y = a + bx + cx^2$	(*parabola*),
(3)	$x = a + by + cy^2$	(*parabola*),
(4)	$y = kx^n$	(*parabolic in form*),
(5)	$xy = c$	(*hyperbola*),
(6)	$y = c10^{kx}$	(*exponential curve*),

we assume that the relation connecting the quantities is the corresponding equation of the above set and it remains to determine the constants of the equation.

If the plotted data does not fit any of the type curves mentioned above, a general method of procedure is to assume an equation of the type

(7) $\qquad y = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n \quad$ (*nth degree curve*).

The coefficients $a_0, a_1, a_2, \cdots, a_n$ can be found from any $n + 1$ pairs of values of x and y.

Since the measurements made in any experiment are liable

to be in error, errors will occur in the computed values of the coefficients. The curve represented by the final equation will not in general pass through the points representing the observed data. Some of these points will be on one side and some on the other. All will be near the curve.

174. **Computation of the Coefficients in the Assumed Formula.** In case the plotted points *appear* to be upon a straight line, a parabola, or a curve of the nth degree, the corresponding equation is assumed and we proceed to determine the coefficients by a method which is illustrated in the following example.

EXAMPLE 1. Let the observed values of x and y be

x	43	85	127	169
y	17	33	49	65

Plotting this data, the points will be seen to lie roughly on a straight line. Hence we assume a relation of the form

$$y = ax + b.$$

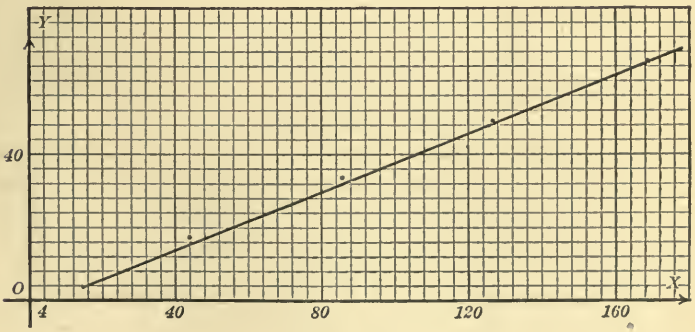


FIG. 125

In this equation replace x and y by their observed values. In this way

we obtain four equations connecting a and b:

$$(8)\qquad \begin{cases} 43a + b = 17, \\ 85a + b = 33, \\ 127a + b = 49, \\ 169a + b = 65. \end{cases}$$

Two equations are necessary and sufficient for the determination of the two unknowns a and b. In general if we have more equations than unknowns the equations are not consistent. That is, the values of a and b as determined from the first two equations are not the same as those obtained from the last two, or from the second and third, etc. Our problem then is to derive from the given set two equations such that the values of a and b obtained therefrom when used as coefficients in the assumed equation will give us a straight line which fits closely the points plotted from the observed data. There are in common use a number of ways of doing this.

First Method. Multiply each equation in turn by the coefficient of a in that equation and add. This gives one equation containing a and b. Multiply each equation in turn by the coefficient of b in that equation and add. This gives a second equation containing a and b. Using the data in (8) above we find in this way the following equations:

$$(9)\qquad \begin{aligned} 53764a + 424b &= 20744, \\ 424a + 4b &= 164. \end{aligned}$$

The solution of these equations for a and b gives

$$(10)\qquad a = 0.39, \qquad b = -0.34$$

Substituting these values of a and b in the assumed equation, we find

$$(11)\qquad y = 0.39x - 0.34.$$

Second Method. When on plotting it is clear that a straight line is the best fitting curve, draw a straight line among the points so that about half are above and half below. The y coördinate of the intersection of this line with the y-axis can then be read directly from the graph and gives the value of b in the equation $y = ax + b$. Measure the angle α that this line makes with the x-axis and then $a = \tan \alpha$.

In case different scales are used on the two axes select two points (x_1, y_1) (x_2, y_2) on the line, then

$$(12)\qquad a = \frac{y_2 - y_1}{x_2 - x_1}.$$

THIRD METHOD. Suppose the best fitting curve is a straight line, *i.e.* that the equation should be of the form

$$y = ax + b.$$

Use for a and b the values obtained on solving the first and last of equations (8). The straight line so found actually passes through the first and last points.

If the points are so distributed that one of the forms (2) or (3) § 173 should be used, proceed to find a, b, and c by using the first, middle, and last points only.

If an equation of degree n [(7), § 173], *i.e.* an equation of the form

$$y = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

should be assumed, use $n + 1$ points evenly distributed along the curve. This method gives us always the same number of equations as there are unknown coefficients to be determined.

FOURTH METHOD. If it is known that the curve is a straight line through the origin then

$$y = kx.$$

Substitute the observed pairs of values of x and y in this equation, add the resulting equations and solve for k. See § 172.

EXERCISES

1. In the following example a series of observed values of y and x are given. The variables are known to be connected by a relation of the form

$$y = ax + b.$$

Find a and b.

Ans. $a = 0.498$, $b = 0.96$

y	6	10.8	16.1	20.6	26
x	10	20	30	40	50

2. The following table gives the density δ of liquid ammonia at various degrees centigrade. Find a relation of the form

$$\delta = at + b.$$

i.e. determine the values of a and b.

t	0	5	10	15
δ	.6364	.6298	.6230	.6160

Ans. $\delta = 0.6364 - 0.0014\,t$

3. The following table gives the specific heat s of hot liquid ammonia at various degrees Fahrenheit. Find a relation of the form $s = at + b$.

t	5	10	15	20	25
s	1.090	1.084	1.078	1.072	1.066

Ans. $s = 1.096 - 0.0012t$

4. In an experiment to determine the coefficient of friction between two surfaces (oak) the following values of F were required to give steady motion to a load W. Plot F and W on squared paper, and find μ where $\mu = F/W$. [CASTLE] *Ans.* $\mu = 3.302$

F	5	10	15	20	25	30	35	40
W	2	3	$4\frac{1}{2}$	$6\frac{1}{4}$	$7\frac{1}{2}$	$9\frac{1}{2}$	$10\frac{1}{2}$	$11\frac{3}{4}$

5. In the following examples a series of values of x and y are given. In each case the variables are connected by an equation of the form $y = ax + b$. Find a and b.

(*a*)

y	5	7.8	11.1	14.2	17
x	9	18	27	36	45

Ans. $a = 0.337$, $b = 1.9$

(*b*)

y	2	3.1	4	5.2	6.2
x	4	8	12	16	20

Ans. $a = 0.2625$, $b = 0.95$

(*c*)

y	5	6.1	8.2	10	12.1
x	1	2	3	4	5

Ans. $a = 1.81$, $b = 2.85$

(*d*)

y	4	7	11	14	17
x	10	20	30	40	50

Ans. $a = 0.33$, $b = 0.7$

In the two following sets of data plot the values of E (Electromotive force) and R (Resistance), and determine an equation of the form

$$E = aR + b.$$

(*e*)

E.....	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
R.....	7.5	18	28	38	49	59	68	80	90	100

(*f*)

E....	3	4.5	6	7.75	9.5	11	12.5	13.5	15
R....	14	28	42	56	70	84	98	112	126

6. A wire under tension is found by experiment to stretch an amount l, in thousandths of an inch, under a tension T, in pounds, as follows:

T.....	10	15	20	25	30
l.....	8	12.5	15.5	20	23

Find a relation of the form $l = kT$ (Hooke's law) which best represents these results.

7. In an experiment with a Weston differential pulley block, the effort E, in pounds, required to raise a load W, in pounds, was found to be as follows:

W.....	10	20	30	40	50	60	70	80	90	100
E.....	$3\frac{1}{4}$	$4\frac{7}{8}$	$6\frac{1}{4}$	$7\frac{1}{2}$	9	$10\frac{1}{2}$	$12\frac{1}{4}$	$13\frac{3}{4}$	15	$16\frac{1}{2}$

Find a relation of the form $E = aW + b$.

8. If θ denotes the melting point (Centigrade) of an alloy of lead and zinc containing x per cent. of lead, it is found that

x.....	40	50	60	70	80	90
θ.....	186	205	226	250	276	304

Find a relation of the form $\theta = a + bx + cx^2$.

9. The readings of a standard gas-meter S and those of a meter T being tested on the same pipe line were found to be

S.....	3,000	3,510	4,022	4,533
T.....	0	500	1,000	1,500

Find a formula of the type $T = aS + b$ which best represents these data. What is the meaning of a? of b?

10. An alloy of tin and lead containing x per cent. of lead melts at the temperature θ (Fahrenheit) given by the values

x	25	50	75
θ	482	370	356

Determine a formula of the type $\theta = a + bx + cx^2$.

11. A restaurant keeper finds that if he has G guests a day his total daily expenditure is E dollars, and his total daily receipts are R dollars. The following numbers are averages obtained from the books

G	210	270	320	360
E	16.7	19.4	21.6	23.4
R	15.8	21.2	26.4	29.8

Find the simple algebraic laws which seem to connect E and R with G. [$R = mG$; $E = aG + b$.] What are the meanings of m, a, and b? Below what value of G does the business cease to be profitable?

12. The following statistics (taken from Bulletin 110, part 1 of the Bureau of Animal Industry, U. S. Dept. of Agriculture) give the changes in average egg production between 1899 and 1907:

Year.	Birds Competing per Year.	Eggs Laid.	Actual Average Production.	Added to Actual Average.	Modified Average Due to Abnormal Conditions.
1899–1900	70	9,545	136.36	0	136.36
1900– 01	85	12,192	143.44	0	143.44
01– 02	48	7,468	155.58	0	155.58
2– 3	147	19,906	135.42	23.73	159.15
3– 4	254	29,947	117.90	11.24	129.14
4– 5	283	37,943	134.07	0	134.07
5– 6	178	24,827	140.14	13.95	154.09
6– 7	187	21,175	113.24	29.53	142.77

With the actual and modified averages in hand we may inquire: what has been the general trend of the mean annual egg production during the period covered by the investigation? The clearest answer to this question may be obtained by plotting the figures in the fourth and sixth columns of the above table, and then striking through each

of the two zigzag lines so obtained the best fitting straight line, as determined by the method of least squares. The equations of the two straight lines are as follows:

$$\textit{actual averages:} \quad y = 148.48 - 3.10x,$$
$$\textit{modified averages:} \quad y = 144.13 + 0.043x.$$

In these equations y represents the mean annual egg production and x the year. The origin for x is at 1898–99. Verify these two equations.

13. The following table, taken from the same bulletin, gives the percentage of the flocks laying (*a*) less than 45 eggs, and (*b*) 195 or more eggs in a year.

Annual Egg Production.	1899–1900.	1900–'01.	'01–'02.	'02–'03.	'03–'04.	'04–'05.	'05–'06.	'06–'07.
Less than 45 in % ..	4.29	1.18	0	1.36	6.70	7.07	0.56	4.81
195 or more in % ...	4.29	10.60	18.75	6.12	0.79	12.71	5.06	0

Plot this data, using years for abscissa and percentages for ordinates, making two curves and find by the method of least squares the best fitting lines.

$$\textit{Poor layers:} \quad y = 1.795 + 0.3225x.$$
$$\textit{Good layers:} \quad y = 11.639 - 0.966x.$$

Interpret the sign of the coefficient of x in each equation, and give the meaning of the constant term in each equation.

175. Substitution. If on plotting the given values of x and y the plotted points are seen to be approximately on a branch of a rectangular hyperbola with vertical and horizontal asymptotes we assume a relation of the form

$$(x - a)(y - b) = c, \tag{14}$$

where (a, b) are the coördinates of the intersection of the asymptotes, and proceed to determine a, b, and c.

In many of the cases in which this form appears both a and b are zero and the equation (14) becomes

$$y = c/x.$$

In some cases a is zero and equation (14) becomes

$$y = b + c/x.$$

There are many curves which resemble closely the curve given by equation (14), but whose equation is somewhat different. In order to determine whether (14) is the best equation to represent the plotted data, obtain from the figure an approximate value of a. In many cases $a = 0$. Make the substitution $1/(x - a) = u$ and plot the new points (u, y). If these are approximately upon a straight line then

$$y = b + cu^*$$

and equation (14), in one of its forms, is the proper relation to assume.

If on plotting the observed values of x and y the plotted points appear to be on a parabola with axis parallel to one of the axes and vertex on that axis then call that axis the y-axis and assume

(15) $$y = a + bx^2.$$

The determination of the coefficients a and b can be reduced to that of finding the coefficients in the linear form

$$y = a + bu,$$

where $u = x^2$. As a check that (15) is the correct form to assume plot pairs of values of u and y. If these points appear to be on a straight line then equation (15) is the correct form to assume.

Example. The distance s, in feet, passed over by a falling body in t seconds is found by experiment to be

s	0	5	16	35	65
t	0	.5	1	1.5	2

Find a law connecting s and t.

*This is sometimes called the *reciprocal curve*.

Upon plotting this data, the points are seen to fall on a parabola with vertex upward and at the origin. This suggests that we assume the relation of the form

$$s = at^2.$$

As a check on this assumption we plot the points (u, s) given in the following table:

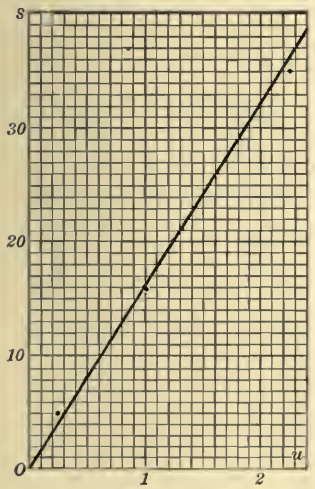


Fig. 126

s	0	5	16	35	65
t	0	.5	1	1.5	2
$u = t^2$.	0	.25	1	2.25	4

These points are approximately upon a straight line $s = au$. The determination of a by the method of least squares gives $a = 16.9$, whence

$$s = 16.9t^2.$$

EXERCISES

1. The following data on the relation of temperature to insect life gives the number of days at a given temperature to complete a given stage of development and is taken from Technical Bulletin, No. 7, Dec. 1913 of the New Hampshire College Ag. Exp. Station.

In each case the plotted points are on a curve of the type $y - b = c/x$ (x = days, y = temperature).

The term *developmental zero* is used to designate that point at which an insect may be kept, theoretically at least, without change for an indefinite period. The developmental zero for the insect and stage approximates the point where the reciprocal curve (calculated from the time factor) intersects the temperature axis. (b = developmental zero.)

For each set, plot the data, and the reciprocal curve; find the developmental zero, and obtain an equation of the form $y - b = c/x$ connecting the data.

(*a*) *Malacosoma americana*, pupal stage. Developmental zero = 11° C.

y.....	32.4	32	26.1	20	16
x.....	9.7	10	13.2	22.5	54

(*b*) *Tenebrio molitor*, incubation of eggs. Developmental zero = 9.5° C.

y.....	31.1	26.6	21	11.6
x.....	6	7.4	12.1	57

(*c*) *Leptinotarsa decemlineata*, incubation of eggs. Developmental zero = 6°.

y.....	32.2	26.7	18.6
x.....	3	3.9	6.3

(*d*) *Toxoptera graminum*, birth to death. Developmental zero = 5°.

y.....	32.5	26.5	21	15.5	10
x.....	10	12	20	30	58

(*e*) Incubation period of eggs of codling moth. Developmental zero = 6°.

y.....	28	25	22	18	16	15
x.....	4.5	6	7	9	12	16

(*f*) *Toxoptera graminum*, birth to maturity. Developmental zero = 5°.

y.....	26.5	21	15.5	10
x.....	6.5	9	15	32

Ans. $y - 5 = \dfrac{150 \text{ (nearly)}}{x}$.

2. Determine a relation of the form $y = a + bx^2$ that best represents the values.

x.....	1	2	3	4	5	6
y.....	14.1	25.2	44.7	71.4	105.6	147.9

3. The pressure p, measured in centimeters of mercury, and the volume v, measured in cubic centimeters, of a gas kept at constant temperature, were found to be as follows.

v.....	145	155	165	178	191
p.....	117.2	109.4	102.4	95	88.6

Determine a relation of the form $pv = k$.

4. Find a formula of the type $u = kv^2$ that best represents the following values.

u.....	3.9	15.1	34.5	61.2	95.5	137.7	187.4
v.....	1	2	3	4	5	6	7

5. If a body slides down an inclined plane, the distance s, in feet, that it moves is connected with the time t, in seconds, after it starts by an equation of the form $s = kt^2$. Find the best value of k consistent with the following data.

s.....	2.6	10.1	23	40.8	63.7
t.....	1	2	3	4	5

Ans. $k = 2.556$.

6. Find approximately the relation between s and t from the following data.

s.....	3.1	13	30.6	50.1	79.5	116.4
t.....	.5	1	1.5	2	2.5	3

176. Logarithmic Plotting. In case the plotted points (x, y) appear to lie on one of the parabolic or hyperbolic curves of the family

(16) $$y = bx^m$$

there is a distinct advantage in taking the logarithm (base 10) of both sides:

(17) $$\log y = m \log x + \log b,$$

and then substitute

(18) X for $\log x$, Y for $\log y$, B for $\log b$

so that the equation (17) becomes,

$$Y = mX + B. \tag{19}$$

If the values of x and y are tabulated in columns, and their logarithms X and Y are looked up and written in parallel columns opposite, then *the points* (X, Y) *should lie on a straight line to justify the assumption of equation* (16). And if they do lie fairly on a line, *its slope and y-intercept determine the constants* m *and* b *of equation* (16). This can often be done graphically from the drawing with sufficient accuracy, but if greater accuracy is required they can be determined from the data by least squares.

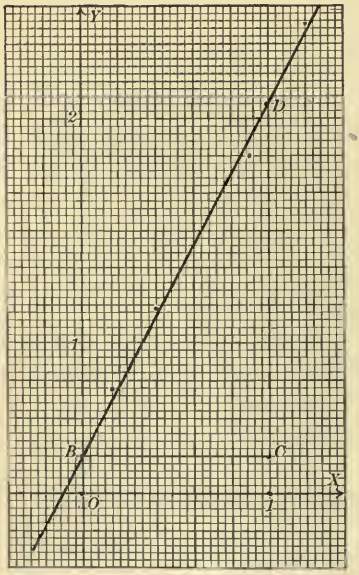


Fig. 127

Example.

x.	y.	$X = \log x$.	$Y = \log y$.
2	6.000	0.3010	0.7782
4	24.60	0.6020	1.3909
8	70.80	0.9030	1.8500
16	338.8	1.2040	2.5299

The points (X, Y) lie nearly on a line BD, Fig. 127. Graphically, we scale off from the figure,

B = the y-intercept OB = 0.2,

m = the slope $= \dfrac{CD}{BC}$

$= \dfrac{47}{25} = 1.88$

By least squares, putting the data into equation (19), we find

$$B = 0.1970 = \log b;$$

hence

$$b = 1.574,$$

$$m = 1.914,$$

and these values in equation (16) give,

$$y = 1.574\, x^{1.914}$$

In case the quantities x and y are connected by a relation of the form

$$y = c10^{kx}, \tag{20}$$

it is advantageous to compute $Y = \log y$ and plot x and Y. If these new values when plotted appear to be on a straight line we write

$$Y = kx + \log c \tag{21}$$

and determine k and $\log c$ by the method of least squares.

177. Logarithmic Paper. Paper, called logarithmic paper, may be bought that is ruled in lines whose distances, horizontally and vertically, from a point O are *proportional to the logarithms* of the numbers 1, 2, 3, etc.

Such paper may be used instead of actually looking up the logarithms in a table. For if the *given values* be plotted on this new paper, the resulting figure is identically the same as that obtained by plotting the *logarithms of the given values* on ordinary squared paper.

The use of logarithmic paper is however not essential; it is merely convenient when one has a large number of problems of this type to solve.

EXERCISES

1. A strong rubber band stretched under a pull of p kg. shows an elongation of E cm. The following values were found in an experiment:

p.....	0.5	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
E.....	0.1	0.3	0.6	0.9	1.3	1.7	2.2	2.7	3.3	3.9

Find a relation of the form $E = kp^n$. *Ans.* $E = .3p^{1.6}$

2. The amount of water A, in cu. ft., that will flow per minute through 100 feet of pipe of diameter d, in inches, with an initial pressure of 50 lbs. per sq. in., is as follows:

d.....	1	1.5	2	3	4	6
A.....	4.88	13.43	27.50	75.13	152.51	409.54

Find a relation of the form $A = kd^n$. *Ans.* $A = 4.88d^{2.473}$

3. In testing a gas engine corresponding values of the pressure p, measured in lbs. per sq. ft., and the volume v, in cubic feet, were obtained as follows:

v	7.14	7.73	8.59
p	54.6	50.7	45.9

Find a relation of the form $p = kv^n$. *Ans.* $p = 387.6v^{-.938}$

4. Find a relation between p and v from the following data:

v	6.27	5.34	3.15
p	20.54	25.79	54.25

Ans. $pv^{1.41} = 273.5$

5. The intercollegiate track records for foot-races are as follows, where d means the distance run, and t the record time:

d	100 yds.	220 yds.	440 yds.	880 yds.	1 mi.	2 mi.
t	$0{:}09\frac{4}{5}$	$0{:}21\frac{1}{5}$	0:48	$1{:}54\frac{4}{5}$	$4{:}15\frac{2}{5}$	$9{:}24\frac{2}{5}$

Find a relation of the form $t = kd^n$. What should be the record time for a race of 1,320 yds.?

6. In each of the following sets of data find a relation of the form $y = kx^n$ connecting the quantities.

(*a*)

v	1	2	3	4	5
p	137.4	62.6	39.6	28.6	22.6

(*b*)

u	12.9	17.1	23.1	28.5	3.0
v	63.0	27	13.8	8.5	6.9

(*c*)

θ	82°	212°	390°	570°	750°	1100°
c	2.09	2.69	2.90	2.98	3.09	3.28

(*d*)

x	1.5	2.5	3.5	4.5	5.5	6.5	7.5	8.5
y	3.05	3.92	4.65	5.30	5.82	6.40	6.85	7.25

Ans. $y = 2.5x^{1/2}$.

7. Draw each of the following curves:

(*a*) $y = x^{1/2}$. (*b*) $y = 2x^2$.
(*c*) $y = 2x^{1/2}$. (*d*) $y = 3x^{3/2}$.
(*e*) $y = 8x^{-3/2}$. (*f*) $y = 1.5x^{2/3}$.
(*g*) $y = 9.2x^{-2/3}$. (*h*) $y = \log x^{2/3}$.
(*i*) $y = 10^x$. (*j*) $y = 2 \cdot 10^{x^2}$.
(*k*) $y = 10^{x/2}$. (*l*) $y = 10^{x+2}$.

8. Find an empirical equation connecting the x and y values given in the following tables.

(*a*)

x	0.2	0.4	0.6	0.8
y	3.18	3.96	5.00	6.30

Ans. $y = 2.51(10^{x/2})$.

(*b*)

x	0.2	0.4	0.6	0.8
y	5.8	4.4	3.4	2.6

(*c*)

x	0	14.4	28.4	42.2
y	180	24	3	0.7

(*d*)

x	0	41.4	83.6	126.2
y	180	92	46	22

9. Given age in years and diameter in inches of a tree $1\frac{1}{2}$ feet from the ground as follows.

Age	19	58	114	140	181	229
Diameter	3	7	13.2	17.9	24.5	33

Plot the data and determine a relation of the form $y = kx^n$.

10. Given age in years and height in feet of a tree as follows:

Age	13	34.4	50.5	218	247
Height	13.4	27.5	38.4	72.5	73

Plot the data and determine a relation of the form $y = kx^n$.

11. Following are vapor pressures, in mm. of mercury, of methyl alcohol at various temperatures:

t.....	6	13	21	30	40
p.....	42	64	100	160	260

Represent these by an empirical formula.

12. The safe load W in tons of 2000 lbs. for a beam 4 inches wide when the distance between the supports is 12 feet is given by

$$W = KD^2,$$

where D is the depth in inches. Find K from the following table:

D.....	10	12	14	16	18
W.....	1.85	2.67	3.63	4.74	6.00

13. Plot a curve from the following data, find its equation, and estimate the price of 36-inch pipe.

Diameter of Sewer Pipe.....	8	10	12	14	16	18	20	22	24
Price in ¢ per linear ft.....	26	27	30	36	50	68	93	125	150

14. Plot a curve from the following data, find its equation, and estimate the pressure for a velocity of 110 miles per hour. The pressure is given in pounds per square foot of cross section of the first car in a train of ten, and the velocity in miles per hour.

v....	32	37	43	48	55	64	68	83	88	91	95
p....	.97	1.35	1.80	2.25	3.32	4.18	4.83	6.75	7.72	8.37	9.01

CHAPTER XII

THE PROGRESSIONS

178. Arithmetic Progression. A sequence of numbers in which each term differs from the preceding one by the same number is called an ***arithmetic progression*** (denoted by ***A. P.***). The ***common difference*** is that number which must be added to any term to obtain the next one.

To determine whether or not a given sequence is an arithmetic progression we find and compare the successive differences of consecutive terms. Thus

$$3, 10, 17, 24, 31, \cdots$$

is an A. P. in which the common difference is 7.

$$5, 8, 11, 15, 18, \cdots$$

is not an A. P.

179. Notation. The following symbols are commonly used to denote five important numbers, called ***elements***, which are considered in connection with arithmetic progressions.

a or a_1 = the first term
n = the number of terms
l or a_n = the last or nth term
d = the common difference
s or s_n = the sum of the first n terms

180. Formulas. If the terms of an arithmetic progression are written down and numbered as follows,

Terms: $a,\ a + d,\ a + 2d,\ a + 3d, \cdots$
Number of term: 1, 2, 3, 4, $\cdots$

we observe that the coefficient of d in each term is one less than the number of the term. Hence for the last or nth term we have

$$l = a + (n - 1)d \tag{1}$$

We may write the progression in which l is the last term as follows:

$$a, \quad a + d, \quad a + 2d, \quad \cdots, \quad l - 2d, \quad l - d, \quad l.$$

The sum of an arithmetic progression is found by adding the n terms together:

$$s = a + (a + d) + (a + 2d) + \cdots + (l - 2d) + (l - d) + l.$$

Inverting the order of the terms

$$s = l + (l - d) + (l - 2d) + \cdots + (a + 2d) + (a + d) + a.$$

By addition of corresponding terms, we have

$$\begin{aligned} 2s &= (a + l) + (a + l) + (a + l) + \cdots + (a + l) + (a + l) \\ &= n(a + l). \end{aligned}$$

It follows that

$$s = \frac{n}{2}(a + l). \tag{2}$$

Example. Find the sum of an arithmetic progression of six terms whose first term is 4 and whose common difference is 2.

Since $n = 6$, we have $l = 4 + 5 \cdot 2 = 14$. Hence $s = \frac{6}{2}(4 + 14) = 54$.

Given any three of the elements a, n, l, d, s, either of the other two can be found by substituting in (1) or (2) and solving. If n is to be found, the given elements must be such that the formula will be satisfied by a positive integral value of n.

Example. Given $d = \frac{1}{2}$, $l = \frac{3}{2}$, $s = -1\frac{5}{2}$; find a and n. Substituting in (1) and (2), we have

$$\frac{3}{2} = a + \frac{1}{2}(n - 1), \qquad -\frac{15}{2} = \frac{1}{2}n\left(a + \frac{3}{2}\right). \tag{3}$$

Eliminating a,

$$n^2 - 7n - 30 = 0.$$

Solving for n,

$$n = 10 \quad \text{or} \quad -3.$$

The value $n = -3$ is inadmissible. Substituting $n = 10$ in (3), we obtain $a = -3$. Hence $n = 10$, $a = -3$, and the arithmetic progression is $-3, -2\frac{1}{2}, -2, -1\frac{1}{2}, -1, -\frac{1}{2}, 0, \frac{1}{2}, 1, 1\frac{1}{2}$.

181. Arithmetic Means. The terms of an arithmetic progression between the first and last terms are called ***arithmetic means.*** Between any two numbers as many arithmetic means as desired can be inserted. To do this we can use equation (1) to compute the common difference d, for a and l are known and n is two more than the number of terms to be inserted. Then the required means are $a + d$, $a + 2d$, etc.

The problem of inserting one arithmetic mean between two numbers is the same as the problem of finding the average of two numbers. If m is the average of a and b, then

$$m = \frac{a + b}{2},$$

and a, m, b form an arithmetic progression. For this reason m is called ***the arithmetic mean*** of a and b.

EXAMPLE. Insert 4 arithmetic means between 7 and 20. Here $a = 7$, $l = 20$, $n = 6$. Substituting these values in (1), we have $20 = 7 + 5 \cdot d$, whence $d = 2\frac{3}{5}$. Hence, the required means are $9\frac{3}{5}$, $12\frac{1}{5}$, $14\frac{4}{5}$, $17\frac{2}{5}$.

EXERCISES

Determine which of the following suites of numbers form arithmetic progressions.

1. $1, 7, 9, 12, \cdots$
2. $x, x^2, 3x, \cdots$
3. $5, 8, 11, 14, \cdots$
4. $a - 2b, a, a + 2b, \cdots$
5. $3, 7, 11, 15, \cdots$
6. $4, 2, 0, -2, \cdots$
7. $2, 4, 6, 9, \cdots$
8. $5, 3, 1, -1, \cdots$

Find l and s for the following progressions:

9. $-2, -6, -10, \cdots$ to 17 terms.
10. $3, 10, 17, \cdots$ to 50 terms.
11. $5, 7.5, 10, \cdots$ to 36 terms.
12. $2, \frac{8}{3}, \frac{10}{3}, 4, \cdots$ to 48 terms.
13. Solve formula (1) for a, n, and d in turn.
14. Solve formula (2) for a, n, and l in turn.
15. Given $n = 20$, $a = 1$, $d = 7$; find l and s.
16. Given $n = 1000$, $l = 500$, $d = \frac{1}{4}$; find a and s.
17. Given $n = 16$, $a = 2$, $l = 3$; find d and s.
18. Given $a = 2$, $l = 3$, $s = 100$; find n and d.
19. Given $n = 9$, $a = 1$, $s = 37$; find d and l.
20. Given $a = 4$, $d = 0.1$, $l = 8$; find n and s.
21. Given $n = 10$, $d = 0.2$, $s = 78$; find a and l.
22. Given $n = 12$, $l = -3$, $s = 140$; find a and d.
23. Given $d = 3$, $l = 22$, $s = 87$; find a and n.
24. Given $a = 8$, $d = 8$, $s = 80$; find l and n.
25. Insert 3 arithmetic means between 1 and 17.
26. Insert 4 arithmetic means between 2 and 18.
27. Insert 5 arithmetic means between 3 and 38.
28. Insert 6 arithmetic means between 4 and 6.
29. Eight stakes are to be set at equal distances between the two corners of a 60 ft. lot. How far apart must they be? *Ans.* 6 ft. 8 in.
30. I desire to close up one side of crib 12 feet 4 inches high, with 6 inch boards. I have just 21 boards. I desire to leave a 1 inch crack at top and bottom. How far apart must I place the boards to have them equally spaced? *Ans.* 1 inch.
31. At the end of each year for 10 years a man invests \$200 on which he collects annual interest at 6%. Find the total interest received. *Ans.* \$540.
32. The population of a certain town has made a net gain of the same number of people each year for the last 30 years. In 1893 it was 1523; in 1906 it was 2212. What was it in 1890? in 1902? in 1916? Predict the population for 1925.
33. What will it cost to erect the steel work of a 20 story building at \$3000 for the first story and \$250 more for each succeeding story than for the one below? *Ans.* \$107500.

34. I drop a rock over a cliff 400 ft high. How long before I hear it strike bottom if it falls 16 ft. the 1st second, 48 ft. the 2d second, 80 ft. the 3d second, etc., and sound travels 1090 ft. per second in air?

Ans. $5\frac{2}{3}$ sec. nearly.

35. A ball rolling down an incline goes 2 ft. the first second and 6 ft., 10 ft., 14 ft., respectively in the next three seconds, starting from rest. How far will it roll in 15 seconds? *Ans.* 450 ft.

36. A clock strikes the hours and also 1, 2, 3, 8, respectively, at the quarter hours. How many strokes does it make in a day? *Ans.* 422.

37. A farmer is building a fence along one side of a quarter section. The post holes are dug one rod apart and the posts are piled at the first. How far will he walk to distribute them one at a time and return to set the first one? *Ans.* $20\frac{1}{4}$ miles.

38. Find the sum of all multiples of 7 less than 1000. *Ans.* 71071.

39. Find two numbers whose arithmetic mean is 11 and the arithmetic mean of their squares is 157.

40. Show that if an A. P. has an odd number of terms the middle term is the arithmetic mean of the first and last.

41. If the sum of any number of terms of the A. P. 8, 16, 24, $\cdots$ be increased by 1, the result is a perfect square.

182. **Geometric Progression.** A sequence of numbers in which each term may be found by multiplying the preceding term by the same number is called a ***geometric progression*** (denoted by ***G. P.***). The constant multiplier is called the ***common ratio.*** Thus

$$3,\ 15,\ 75,\ 375,\ \cdots$$

is a G. P. in which the common ratio is 5.

The elements of a geometric progression are the first term a or a_1, the number of terms n, the last or nth term l or a_n, and the sum s or s_n of the first n terms.

183. **Formulas.** If the terms of a geometric progression be written down and numbered as follows,

Term:	$a,\ ar,\ ar^2,\ ar^3,\ \cdots$
Number of term:	1, 2, 3, 4, $\cdots$

we see that the exponent of r in each term is one less than the number of the term. Hence for the nth or last term we have

$$l = ar^{n-1} \tag{4}$$

The sum of the first n terms of the preceding geometric progression is

$$s = a + ar + ar^2 + \cdots + ar^{n-1}$$

Multiplying both sides by r,

$$sr = ar + ar^2 + ar^3 + \cdots + ar^n$$

By subtraction, we have

$$sr - s = ar^n - a.$$

Solving the last equation for s, we get

$$s = \frac{a(r^n - 1)}{r - 1} = \frac{a(1 - r^n)}{1 - r}. \tag{5}$$

From (4) we obtain $rl = ar^n$. Hence (5) may also be written

$$s = \frac{a - rl}{1 - r}. \tag{6}$$

The two fundamental formulas (4) and (6) contain the five elements a, l, n, r, s, any two of which may be found if the other three are given.

EXAMPLE 1. Find s if $a = 1$, $n = 7$, $r = 4$.
Substituting these values in (5), we get

$$s = \frac{4^7 - 1}{4 - 1} = \frac{16384 - 1}{3} = 5461.$$

184. Geometric Means. If three positive numbers are in geometric progression the middle one is said to be the ***geometric mean*** of the other two. It is easy to see that *the geometric mean of two numbers is the square root of their product.* Thus 3 is the geometric mean of $2\frac{1}{4}$ and 4.

If several numbers are in geometric progression all the inter-

mediate terms are said to be geometric means between the first and last terms. We can insert as many geometric means as we wish between any two positive numbers. To do this we use equation (4), § 183, to compute r; a, l, and n being known. Then the desired means are ar, ar^2, ar^3, etc.

EXAMPLE. Insert three geometric means between 4 and 16. Since 16 is to be the 5th term we have $a = 4$, $ar^4 = 16$, whence $r^4 = 4$ and $r = \sqrt{2}$; hence the five terms are $4, 4\sqrt{2}, 8, 8\sqrt{2}, 16$.

EXERCISES

Which of the following sets of numbers form geometric progressions?

1. $3, -6, 12, -24, \cdots$
2. $4, 6, 9, 13.5, \cdots$
3. $7, 18, 40, \cdots$
4. $8, 12, 18, 26, \cdots$
5. $a, 2a, 3a, 4a, \cdots$
6. $a, a^2, a^3, \cdots$
7. $\sqrt{3} - 1, \sqrt{2}, \sqrt{3} + 1, \cdots$
8. $8, 4, 2, 1, \cdots$
9. $a, -a^2, a^3, -a^4, \cdots$
10. $\sqrt{2}, 2, 2\sqrt{2}, 4, \cdots$
11. $\sqrt{2}, \sqrt{6}, 3\sqrt{2}, \cdots$
12. $9, 3, 1, \frac{1}{3}, \cdots$
13. Solve formula (4) for a, n, and r in turn.
14. Solve formula (6) for a, l, and r in turn.
15. Given $a = 2$, $r = 3$, $n = 12$; find l and s.
16. Given $a = 3$, $r = 5$, $n = 10$; find l and s.
17. Given $a = 4$, $n = 6$, $s = 252$; find l and r.
18. $l = 486$, $a = 2$, $n = 6$; find r and s.
19. Given $a = 15$, $r = 3$, $l = 3645$; find n and s.
20. Given $n = 5$, $r = \frac{1}{4}$, $l = 512$; find a and s.
21. Insert two geometric means between 2 and 128.
22. Insert 3 geometric means between 2 and 162.
23. Insert 2 geometric means between $\sqrt{2}$ and 108.
24. What is the geometric mean between a/b and b/a?
25. Find the 6th term and the sum of the series $2, 4, 8, \cdots$.
26. It takes 32 nails to shoe a horse. A blacksmith agrees to drive them as follows: 2 cents for the first, 4 cents for the second, 8 cents for the third, etc. What is the total cost? *Ans.* \$85,899,345.90
27. Find the amount of \$500 in 5 years at 6% compounded annually; compounded semiannually. *Ans.* \$669.10; \$672.45

28. In how many years will \$100 amount to \$200, interest at 8% compounded annually? In how many years with interest at 6% compounded annually?

Ans. 9 years approximately; 12 years approximately.

29. A man promises to pay \$10,000 at the end of 5 yr. What amount must be invested each year at 6% compound interest so that at the end of the time the debt can be paid?

30. A premium of \$104 is paid to an insurance company each year for 10 years.

What is the value of these amounts at the end of the time if accumulated at 3% compound interest?

31. A premium of \$91 is paid to an insurance company each year for 10 years.

What is the value of these amounts at the end of the time if accumulated at 3% compound interest?

What is the value if accumulated at 4% compound interest?

32. An insurance company agrees to pay me \$1000 a year for 10 years, or an equivalent cash sum to myself or heirs at the end of the period.

Compute the equivalent cash sum if money is worth 6% compound interest.

33. A father invests \$100 each year for a newborn son, beginning when he is one year old.

If money is worth 4% compounded annually, what sum is due the son on his twenty-first birthday?

What does he receive on his twenty-first birthday if the amounts invested bear 3% compound interest?

34. A potato cuts into 4 parts for planting, each piece produces 5 good sized potatoes, 80 of which make a bushel. If I plant each year all that I raised the preceding year, how many bushels of potatoes will I have at the end of the fifth year? How much are they worth at \$4.00 per bu.? *Ans.* \$160,000.

35. One kernel of corn planted produces a stalk with 2 ears with 16 rows each, 50 kernels to the row. Suppose 100 ears make a bushel and that I plant each year one-half of all that I raised the preceding year and that one-half of the kernels grew and produced. How many

bushels would I have at the end of the fifth year? (Assume two kernels planted the first year.)

36. I have one sow. Let us suppose that the average litter of pigs is 6, sexes equally distributed, and that I keep all of the sows each year but sell all the others. How many sows in the sixth generation? How many pigs will have been sold after I have disposed of 1/2 of the last or 5th litter? *Ans.* 243; 363.

37. The common housefly matures and incubates a new litter every 3 weeks. There are approximately 200 to a litter evenly distributed as to sex. What will be the number of descendents of one female fly in 12 weeks? *Ans.* 2×10^8.

38. Grasshoppers hatch yearly a brood of 100 evenly distributed as to sex. Assuming that none are destroyed, what will be the number of descendants of one female grasshopper at the end of 5 years? 6 years?

39. The apple aphis matures and incubates in 10 days. The progeny, all females, are 5 in number. The female propagates 5 each day for 30 days. What will be the number of descendants of one female at the end of 30 days?

40. If the population doubled every 40 years, how many descendants would one person have after 800 years? *Ans.* 1,048,576.

41. Find the amount of money that could profitably be expended for an overcoat which lasts 5 years provided it saved an annual doctor bill of \$5, money being worth 6% compound interest.

42. The effective heritage contributed by each generation and by each separate ancestor according to the law of ancestral heredity as stated by Galton is shown in the following table from Davenport.

Generation Backward.	Effective Contribution of Each Generation.	Number of Ancestors Involved.	Effective Contribution of Each Ancestor.
1.....	1/2	2	1/4
2.....	1/4	4	1/16
3.....	1/8	8	1/64
4.....	1/16	16	1/256
5.....	1/32	32	1/1024

Compute the effective contribution of the last 20 generations. The number of ancestors involved in the 20th generation backward and the total number of ancestors involved. The effective contribution of each ancestor in the 20th generation backward.

185. **Infinite Geometric Series.** A geometric progression can be extended to as many terms as we please, since on multiplying any term by the common ratio we obtain the next one. Any series which has no last term and can be indefinitely extended is called an infinite series.

Suppose the terms of a geometric series are all positive. If we begin at the first and add term after term the sum always increases. If $r > 1$, this sum becomes infinite, *i.e.*, if we choose a positive number N no matter how large it is possible to add terms enough that the sum will exceed N. If however $r < 1$, the case is quite different. The sum does not become infinite; it converges to a limit, *i.e.*, it is possible to find a number L such that the sum will exceed any number whatever less than L, *but it will never reach L.* For example the sum obtained by adding terms of the geometric series

$$1 + \tfrac{1}{3} + \tfrac{1}{9} + \tfrac{1}{27} + \cdots$$

in which $r = \frac{1}{3}$, will never reach 1.5, but terms enough can be added to make the sum exceed any number less than 1.5. If, *e.g.*, we wish to make the sum greater than 1.49, five terms are sufficient.

A geometric series in which $r < 1$ is called a decreasing geometric series. *The limit to which the sum of the first n terms of a decreasing geometric series converges is* $a/(1 - r)$, *i.e., the first term divided by one minus the ratio.*

For by (5) § 183,

$$s_n = \frac{a(1 - r^n)}{1 - r} = \frac{a}{1 - r} - \frac{a}{1 - r} \cdot r^n.$$

Now as we add more and more terms, the n in this formula gets larger and larger, a and r remain fixed. Since $r < 1$, it follows that $r^2 < r$, $r^3 < r^2$, etc., and r^n converges to zero when n is taken larger and larger. Therefore the second term on the right converges to zero, and s_n converges to $a/(1 - r)$. This

limit is sometimes called the " sum " (although strictly it is not a sum) of the infinite decreasing geometric series

$$a + ar + ar^2 + \cdots,$$

and we write

$$s = \frac{a}{1 - r}, \qquad r < 1. \tag{7}$$

EXAMPLE. The repeating decimal .666 ··· can be written thus $.6 + .06 + .006 + \cdots$. It is therefore an infinite geometric series whose first term is .6 and whose common ratio is .1. Hence

$$s = \frac{.6}{1 - .1} = \frac{2}{3}.$$

EXERCISES

Find the sum of the following infinite series:

1. $1 + 0.5 + 0.25 + \cdots$.
2. $1 - 0.5 + 0.25 - 0.125 + \cdots$.
3. $1 + \frac{1}{3} + \frac{1}{9} + \cdots$.
4. $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \cdots$.
5. $1 + \frac{2}{3} + \frac{4}{9} + \cdots$.
6. $1 + \frac{5}{6} + \frac{25}{36} + \cdots$.
7. $3 + \frac{3}{4} + \frac{3}{16} + \cdots$.
8. $100 + 1 + 0.01 + \cdots$.
9. $3 + 0.3 + 0.03 + \cdots$.
10. $0.23 + 0.023 + 0.0023 + \cdots$.

Find the value of the following repeating decimals:

11. .1111 ···.
12. .2222 ···.
13. .252252 ···.
14. 1.2424 ···.
15. 2.53131 ···.
16. 2.3452345 ···.
17. .00032525 ···.
18. .1234512345 ···.
19. 20.2020 ···.
20. 5.312312 ···.
21. 6.4141 ···.
22. 3.214214 ···.

CHAPTER XIII

ANNUITIES*

186. Definitions. Suppose you take out a life insurance policy on which you agree to pay a premium of \$100 at the end of each year for 10 years. Such an annual payment of money for a stated time is termed an ***annuity***. Instead of paying \$100 a year you may prefer to pay \$24 at the end of every three months or \$206 at the end of every two years. In any case the stated amount paid at the end of equal intervals of time is called an annuity.

Suppose the stated sums are not paid when due and that after the lapse of say 5 years you desire to pay off your indebtedness with interest compounded. The sum due is called the ***amount*** of the annuity for the five years.

Suppose you buy a house and agree to pay \$1000 at the end of each year for 4 years. This is an annuity. An equivalent cash price at the time of sale is called the ***present value*** of the annuity.

187. Notation. The letter r stands for the ***rate of interest,*** *e.g.* 6; the letter i $(= r/100)$ stands for the ***annual interest*** on one dollar, *e.g.* .06.

The symbol $S_{\overline{n}|}$ stands for the ***amount of an annuity*** of one dollar *paid at the end of each year* for n years.

* A more extended treatment of this subject using substantially the same notation as in this book is to be found in books on investment, insurance, etc., *e.g.* SKINNER, *Mathematical Theory of Investment*, Ginn and Co., to which the authors of this book are indebted for many ideas, methods, and exercises.

The symbol $S^{(p)}_{\overline{n}|}$ stands for the ***amount of an annuity*** of one dollar *paid at the end of each pth part of a year* for n years.

The symbol $a_{\overline{n}|}$ stands for the ***present value*** of one dollar *paid at the end of each year* for n years.

The symbol $a^{(p)}_{\overline{n}|}$ stands for the ***present value of an annuity*** of one dollar *paid at the end of each pth part of a year* for n years.

188. Amount of an Annuity. It is sufficient to consider an annuity of one dollar since the amount for any other sum will be proportional to this.

The first payment of one dollar made at the end of the first year will bear interest for $n - 1$ years, and at the end of the period the amount due will be $(1 + i)^{n-1}$. The second payment will bear interest for $n - 2$ years and will increase to $(1 + i)^{n-2}$. The next to the last payment will bear interest for one year and will increase to $1 + i$. The last payment will be one dollar and it will bear no interest. The total amount $S_{\overline{n}|}$, due at the end of n years is therefore

$$1 + (1 + i) + (1 + i)^2 + \cdots + (1 + i)^{n-2} + (1 + i)^{n-1}.$$

In this geometric progression the first term is 1, the last term is $(1 + i)^{n-1}$, and the ratio is $1 + i$. Substituting these values in the formula (5) § 183 for the sum of a geometric progression, we find

$$(1) \qquad S_{\overline{n}|} = \frac{(1 + i)^n - 1}{i}.$$

189. Partial Payments. Suppose that the payments instead of being made at the end of each year are made at the end of *each pth part of a year for n years.* Consider an annuity of one dollar.

The payment to be made at each payment period is $1/p$. The first payment will bear interest for $n - 1/p$ years. The second payment will bear interest for $n - 2/p$ years, and so on. The next to the last payment will bear interest for $1/p$ years. The

last payment will bear no interest. The total amount due is then

$$\frac{1}{p}+\frac{1}{p}(1+i)^{\frac{1}{p}}+\frac{1}{p}(1+i)^{\frac{2}{p}}+\cdots+\frac{1}{p}(1+i)^{n-\frac{1}{p}}.$$

In this geometric progression the common ratio is $(1+i)^{\frac{1}{p}}$, and by (5), § 183, the sum of the terms is

$$S^{(p)}_{\overline{n}|}=\frac{1}{p}\frac{(1+i)^n-1}{\sqrt[p]{1+i}-1}. \tag{2}$$

As shown in § 145 for the square root, the pth root of $1+i$ is nearly equal to $1+i/p$. In fact it is customary in computing the amount of one dollar at interest compounded p times a year, to use $1+i/p$ instead of $\sqrt[p]{1+i}$. See § 217. If this approximate value be used in formula (2), the right member reduces to

$$\frac{(1+i)^n-1}{i}=S_{\overline{n}|},$$

which shows that $S^{p}_{\overline{n}|}$ is approximately equal to $S_{\overline{n}|}$.

EXERCISES

1. Find the amount of an annuity of \$200 for 10 years at 3%; 4%; 5%; 6%; 8%. *Ans.* For 3% \$2292.78

2. The semiannual premium on an insurance policy is \$50. Find the amount of this annuity for 10 years at 4%. *Ans.* \$606.37

3. The quarterly premium on a policy is \$62.10. Find the amount of this annuity for 10 years at 3%. *Ans.* \$719.11

4. The annual rent of a house is \$480. Find the amount of this annuity for 20 years at 6%. Find the amount if the rent is paid monthly. *Ans.* \$17657.08

5. A man saves and at the end of each year for 40 years deposits \$100 in a savings bank which pays 4% compounded annually. Find the amount. *Ans.* \$9502.55

6. A man saves \$500 a year and invests savings and interest in bonds yielding 6%. What will his accumulations amount to in 10, 15, 20, 30 years? *Ans.* \$6590.40

190. Given the Amount of an Annuity to find the Annuity. Let the annual payment be x. The first payment made one year from the beginning of the term of the annuity will bear interest for $n - 1$ years and will increase to $x(1 + i)^{n-1}$. Likewise, the second will increase to $x(1 + i)^{n-2}$, the third to $(1 + i)^{n-3}$, and so on, while the last payment x will bear no interest. If the sum of the amounts due at the end of n years is \$1, we have

$$x[(1 + i)^{n-1} + (1 + i)^{n-2} + \cdots + (1 + i) + i] = 1.$$

The expression within the square brackets is a geometric progression of n terms with ratio $(1 + i)$; hence, by (5), § 183, we have

$$x\frac{(1 + i)^n - 1}{i} = 1,$$

or

$$x = \frac{i}{(1 + i)^n} - 1, \tag{3}$$

which gives the annuity whose amount after n years is \$1. This formula for x may be written symbolically in the form

$$x = \frac{1}{S_{\overline{n}|}}. \tag{4}$$

EXERCISES

1. In 10 years a man desires to be worth \$30,000. What sum must he set aside yearly to realize that amount if money is worth 8%?

2. An auto truck costing \$2000 lasts 5 years. What sum must be set aside annually at 6% to replace the truck when worn out?

3. An automobile costs \$1500 and lasts 5 years. What is the equivalent annual expenditure, money worth 6%?

4. A city decides to pave some of its streets. For this purpose bonds, bearing 6% interest, to the amount of \$50,000 are issued. The bonds are due in 10 years. What sum must be collected yearly in taxes and invested at 6% to pay off the bonds when due?

191. Present Value of an Annuity. The present value of

one dollar due in one year is $(1 + i)^{-1}$,
one dollar due in two years is $(1 + i)^{-2}$,
. . . . . .
one dollar due in n years is $(1 + i)^{-n}$.

The present value of one dollar paid at the end of each year for n years will then be

$$(1 + i)^{-1} + (1 + i)^{-2} + \cdots + (1 + i)^{-n}.$$

The sum of this geometric progression is the present value sought. Hence the present value, $a_{\overline{n}|}$, of an annuity of \$1 is

$$a_{\overline{n}|} = \frac{(1 + i)^{-n-1} - (1 + i)^{-1}}{(1 + i)^{-1} - 1}. \tag{5}$$

Multiplying numerator and denominator by $1 + i$ we find

$$a_{\overline{n}|} = \frac{1}{i}\left[1 - \frac{1}{(1 + i)^n}\right]. \tag{6}$$

EXERCISES

1. A man buys a farm, agreeing to pay \$1500 cash and \$1500 at the end of each year for three years. What would be the equivalent cash value of the farm if money is worth 6%?

2. A man buys a farm, agreeing to pay \$2000 cash and \$2000 at the end of each year for ten years. What would be the equivalent cash value of the farm if money is worth 6%?

3. A contractor performs a piece of work for a city and takes bonds in payment. The bonds do not bear interest, and are payable in 10 equal annual installments of \$2000, the first payment to be made one year from date. Money being worth 6%, payable annually, what is the cash value of the bonds on the date of issue?

4. Prove that the present value of one dollar paid at the end of each pth part of a year for n years is

$$\frac{1 + \dfrac{1}{(1 + i)^n}}{p(\sqrt[p]{1 + i} - 1)}$$

and show that this is approximately equal to $a_{\overline{n}|}$. See § 189.

5. A man contracts to buy a house paying \$200 every three months for 8 years. Find the equivalent cash price, money being worth 6%.

6. Find the cash value of semiannual payments of \$500 for 5 years, money being worth 6%.

192. Cost of an Annuity. A man desires to provide for his family, in event of his death, an annuity of \$5000 a year for 20 years. What amount must he set aside in his will to provide for this annuity, assuming that money is worth 6%.

The cost of an annuity of *one dollar* per year for n years is $A_{\overline{n}|}$, §§ 187, 191. Whence the cost C, of an annuity of P dollars per year for n years is

$$C = \frac{P}{i}\left[1 - \frac{1}{(1+i)^n}\right]. \tag{7}$$

From this we compute that the man should set aside in his will about \$57350.

EXERCISES

1. What will be the cost of an annuity of \$500 a year for 10 years, money being worth 4%? *Ans.* \$4055

2. A man agrees to pay \$700 a year for 5 years for a house. What is the cash value of the house, money being worth 6%. *Ans.* \$2948.66

3. A man agrees to pay \$700 a year for 20 years for a farm. What is the cash value of the farm, money being worth 5%? *Ans.* \$8723.55

4. A man 70 years old has \$3000. His expectation of life being 8 years, what annuity can an insurance company offer him, money being worth 4%?

5. A man with \$10,000 pays it into a life insurance company which agrees to pay him or his heirs a stated sum each year for 20 years. What is the yearly payment, money being worth 4%?

6. A man buys a house for \$4000. What annual payment will cancel the debt in 5 years, money being worth 6%? *Ans.* \$949.60

7. How long will it take a man to accumulate \$100,000, by saving \$1000 a year and investing it at 6%. *Ans.* 33 yrs.

8. A man inherits \$20,000 which is invested at 4%. If \$1000 a year is spent, how long will the inheritance last. *Ans.* 41 yrs.

193. **Perpetuities.** In the previous problems treated in this chapter the payments continued over a fixed number of years and then stopped. The annual amount expended for repairs on a gravel road does not stop at the end of a given period, but continues forever. Such payments constitute an endless annuity, which is called a ***perpetuity***. Other examples are the annual repairs on a house, taxes, annual wage for a flag man, annual pay of a section gang. The *amount* of an annuity would evidently increase indefinitely as time went on. The *present value* of a perpetuity, however, has a definite meaning. The present value of a perpetuity is a sum which put at interest at the given rate will produce the specified annual income forever. Denote by V the present value of the perpetuity and by P the annual payment. Then

$$V \cdot i = P. \tag{8}$$

If the payments are made every n years instead of yearly, the present value of the perpetuity is denoted by V_n; its value will be

$$V_n = P[(1+i)^{-n} + (1+i)^{-2n} + \cdots + (1+i)^{-pn} + \cdots]. \tag{9}$$

This is an infinite geometric progression whose first term is $P(1+i)^{-n}$ and whose ratio is $(1+i)^{-n}$. Hence, by (7), § 185, the present value of the perpetuity is

$$V_n = P\frac{(1+i)^{-n}}{1-(1+i)^{-n}} = \frac{P}{(1+i)^n - 1}. \tag{10}$$

EXERCISES

1. What is the present cash value of a perpetual income of \$1200 per year, money being worth 6%? *Ans.* \$20,000.

2. How much money must be invested at 6% to provide for an indefinite number of yearly renewals of an article costing \$24?

3. How much money must be invested at 4% to provide for the purchase every 4 years of a \$1000 truck?

4. What is the cash value of a farm that yields an average annual profit of \$2400, money being worth 6%?

5. The life of a certain farming implement costing \$100 is 6 yrs. Find what sum must be set aside to provide for an indefinite number of renewals, money being worth 4%.

6. The life of a University building costing \$100,000 is 100 years. A man desires to will the University enough money to erect the building and to provide for an indefinite number of renewals. How much must he leave the institution?

CHAPTER XIV

AVERAGES *

194. Meaning of an Average. In referring to a group of individuals, a detailed statement of the height of each would take considerable time, when large numbers are involved. In comparing two or more groups, such a mass of detail might fail to leave a definite impression as to their relative heights. What is needed is a single number, between that of the shortest and that of the tallest, which is representative of the group with respect to the character measured. Such an intermediate number is called an ***average.***

The idea of an average is in use in everyday affairs. We hear mentioned frequently such expressions as the average rainfall, the average weight of a bunch of hogs, the average yield of wheat per acre for a county or state, the average wage, the average length of ears of corn, the average increase in population, etc. Often these expressions are used with only an indefinite idea as to what is really meant.

In this Chapter we shall discuss some of the averages in common use, and we shall explain the circumstances under which each is to be used.

195. Arithmetic Average. The ***arithmetic average*** is the

* The authors of this book are indebted for many ideas in this Chapter and for some of its methods to an *Appendix* by H. L. Rietz to E. Davenport, *Principles of Breeding*, Ginn and Co. Some use has been made also of Zizek, *Statistical Averages*, Henry Holt and Co.; Pearson, *Grammar of Science;* Bowley, *Elements of Statistics;* and Secrist, *Introduction to Statistical Methods*, Macmillan.

number obtained by dividing the sum of the measurements taken by the number of those measurements:

$$(1) \qquad \text{arithmetic average} = \frac{\text{sum of all measurements}}{\text{number of measurements}}.$$

Thus, if we measure seven ears of corn and find their lengths to be 6, 7, 8, 9, 10, 11, 12 inches, the arithmetic average of their lengths is 9 inches. Again, the arithmetic average of 6, 7, 8, 12, 12 is 9. This example shows that the arithmetic average gives no indication of the distribution of the items and that there may be no item whose measurement coincides with the average. However, it is influenced by each of the items, and it is easily understood and computed. It should seldom be used except in conjunction with other forms of averages. When used alone it should be for descriptive purposes only.

196. Weighted Arithmetic Average. In measuring the given items it frequently happens that there are

n_1 items with the same measurement l_1,
n_2 items with the same measurement l_2,
. . . . . .
n_k items with the same measurement l_k.

Then the ***weighted arithmetic average*** is given by the formula

$$(2) \qquad \text{weighted arithmetic average} = \frac{n_1 l_1 + n_2 l_2 + \cdots + n_k l_k}{n_1 + n_2 + \cdots + n_k}.$$

In the simple case mentioned above, the weighted arithmetic average gives the same result as the arithmetic average. Its chief advantage is that it facilitates computations. For example the average length of the ears of corn whose individual lengths are 6, 7, 8, 12, 12 can be found as follows:

$$\text{average length} = \frac{1 \times 6 + 1 \times 7 + 1 \times 8 + 2 \times 12}{1 + 1 + 1 + 2} = 9.$$

There may be other reasons, however, for counting one item

several times. Thus, in measurements, an item that is known to be particularly trustworthy may be counted doubly or triply. In such cases, the weighted average differs from the arithmetic average.

197. **The Median.** If we arrange the numbers representing the measurements of the items in order of magnitude, the middle number is called the ***median.*** Thus, the median length of the ears of corn whose lengths are 6, 7, 8, 12, 12 inches is 8 inches. In case there are an even number of items the median is midway between the two middle terms. Thus if the lengths of four ears of corn are 6, 7, 9, 10 inches, the median length is 8 inches. There is no ear of this length among those measured.

The median is often used because it is so easily found. Like the arithmetic mean, it gives no indication of the distribution. It can be used even when a numerical measure is not attached to the various items. For example, ears of corn can be arranged in order of length without knowing the numerical length of any ear; clerks can be ranked in order of excellence; shades of gray may be arranged with respect to darkness of color; etc. The median is the central one of a group and is unaffected by the relative order of the other members of the group. Thus it is used when the primary interest is in the central members.

198. **The Mode.** In measuring the items of a given set it may happen that some one measurement occurs more frequently than any other. This measurement is called the ***mode.*** Thus, the modal length of six ears of corn whose lengths are 6, 7, 8, 12, 12, 13 inches is 12 inches. A set of measurements may have more than one mode. Thus in a given factory there might be few men who received \$2 per day, a large number who received \$3, a small number who received \$4, and a large number who received \$5, while few received more than \$5. There would then be *two* modes for wages, namely \$3, and \$5.

If a curve be plotted using measurements as abscissas and the number of items corresponding to each frequency as ordinates, the mode corresponds to the maximum ordinate or ordinates. (See § 225.)

Unlike the arithmetic average, and the median, the mode is always the value of one individual measurement. Extreme measurements have no effect upon it.

In measuring heights of men we might place all those over 4.5 and under 5.5 feet at 5 feet. For this distribution the mode would necessarily fall at one of the integers. If we arrange the heights in three-inch intervals the mode might not appear as an integer, although it would be near the mode first obtained. Thus it is seen that the mode depends upon the grouping of the measurements.

The existence of a mode shows the existence of a type. It is the mode that we have in mind when we speak of the average height of a three-year-old apple tree, the average price of land, or the average interest rate.

199. The Geometric Average. The geometric mean of two positive numbers has been defined in § 184. By analogy we may define the ***geometric average*** of n positive numbers as the nth root of their product.

If a growing tree doubles its diameter in 20 years what is its annual percentage rate of increase? It is not 5%, for an increase of 5% a year would give the following diameters at the end of the

1st,	2d,	3d, . . ., 20th year
$(1.05)d$,	$(1.05)^2d$,	$(1.05)^3d$, . . ., $(1.05)^{20}d$

which would give a final diameter greater than $2.6d$. Evidently what is wanted is a rate r such that

$$\left(1 + \frac{r}{100}\right)^{20} = 2,$$

whence $r = 100(\sqrt[20]{2} - 1) = 3.53^{+}$. Hence an annual increase

of about $3\frac{1}{2}\%$ will double anything in 20 years. The geometric average is used in many practical affairs. Knowing the average rate of growth of a city in the past the geometric average is used to predict its future growth. When a new school building is being designed, for example, it should be made large enough to meet the future growth of the community as shown by this geometric average.

200. **Conclusion.** Given a set of items numerically measured or not, we should first determine whether or not the data is such as to warrant any kind of an average. Then the decision whether one or another kind of average is to be employed depends upon the use to which the result is to be put. If the data is not complete, the arithmetic average cannot be used. If we desire to characterize a type in such a case, we may find the mode, for which the data need not be complete.

Frequently it is best to make use of more than one kind of average in describing a distribution. It must be remembered that any average at best conveys only a general notion and never contains as much information as the detailed items which it represents.

EXERCISES

1. From the heights of the members of your class, find each of the following kinds of average height: (*a*) arithmetic, (*b*) median, (*c*) mode.

2. Determine in the following cases which average is meant: mean daily temperature; average student; average price of butter; average of a flock with respect to egg production; average salary for all of the teachers of a state; average number of bushels of corn per acre for a state or nation; normal rainfall; average number of pigs per litter; average number of hours of sunshine per day; average speed of train between two stops; average wind velocity; mean annual rainfall; average sized apple; average price of oranges when arranged according to sizes; average date of the last killing frost in the spring; average price of land per acre in a given locality; average gain in weight per day of a hog.

3. What kind of an average is meant in each of the following cases: one fly lays on an average 120 eggs; 63% of the food of bobolinks is insects; every sparrow on the farm eats ¼ oz. of weed seed every day; the average gas bill is $2 per month; the average price received for lots in a subdivision was $800; repairs, taxes, and insurance on a house average $100 per year; the average amount of material for a dress pattern is 8 yards, 36 inches wide; a college graduate earns on an average $1125 a year, while the average yearly earnings of a day laborer, who has no more than completed the elementary school, is $475.

4. Suppose that we consider 5 millionaires and 1000 persons who are in poverty. Find the arithmetic average, the median, and the mode of the wealth of this group. Which best portrays conditions?

5. In the *Christian Herald* for March 10, 1915, p. 237, it is stated that: "The average salary of ministers of all denominations is $663. The few large salaries bring up the average." Which average is used here? Is it the best to portray conditions? Is the result too high or too low to represent conditions properly?

6. Compute for the members of your family the mean age, and arithmetic average. Is there a mode?

7. On a given street ascertain the number of houses per block for 5 blocks. Find the arithmetic average and the median. Is there a mode?

8. On a given business street ascertain the number of stories of each business house for one block. Find the arithmetic average and the median. Is there a mode?

9. Proceed as in Ex. 8 for a residence street. Is there a mode?

10. In 4 years the number of motorists killed at railroad crossings doubled. Find the annual rate of increase, using the geometric average.

Ans. 19%.

11. If in the last 20 years the number of deaths in the U. S. due to consumption has increased 50%, find the annual rate of increase, using the geometric average. *Ans.* 2%.

12. Land increased in value from $40 to $150 per acre from 1890 to 1915. What was the average yearly increase?

13. Find the average (arithmetic) word, sentence, and paragraph length, of some one of the writings of Longfellow, Holmes, Whittier, Poe; of some short story; of some newspaper article.

14. The total of the future years which will be lived by 100,000

persons born on the same day are 5,023,371. If the total number of years to be lived is divided by the number of persons the quotient will be the average number of future years to be lived by each person. What kind of an average is this? What average age does it give?

15. Out of 100,000 males born alive on the same date about one-half, namely 50,435, attain age 59. This is then an average age attained. What kind of an average is it?

CHAPTER XV

PERMUTATIONS AND COMBINATIONS

201. Introduction. In how many ways can I make a selection of two men to do a day's work if there are 3 men available for the forenoon and 4 for the afternoon? Having hired one man for the forenoon, I can hire any one of 4 for the afternoon, and since this is true for each of the three, there are $3 \times 4 = 12$ ways of making the selection. This reasoning is general; that is, it does not depend upon the special properties of the numbers 3 and 4. Hence we see that *if there are p ways of doing a first act, and if corresponding to each of these p ways there are q ways of doing a second act, then there are pq ways of doing the sequence of two acts in that order.*

It is evident also that this principle applies to a sequence of more than two acts and we may say,

If there are p ways of doing a first act; and if after this has been done in any one of these p ways there are q ways of doing a second act; etc.; and if after all but the last of the sequence have been done there are r ways of doing the last act, then all the acts of the sequence can be done in the given order in pq ⋯ r ways.

EXERCISES

1. With 4 acids and 6 bases, how many salts can a student make?
2. A ranchman has 5 teams, 4 drivers, and 3 wagons. In how many ways can he make up one outfit?
3. There are 6 routes from Chicago to Seattle, 4 from Seattle to Portland, 3 from Portland to San Francisco. How many ways are there of going from Chicago to San Francisco via Seattle and Portland?

202. **Combinations and Permutations.** A group of things selected from a larger group is called a ***combination.*** The things which constitute the group are called ***elements.*** Two combinations are alike if each contain all the elements of the other irrespective of the order in which they appear. Two combinations are different if either contains at least one element not in the other.

A ***permutation*** of the elements of a group or combination, or simply a *permutation,* is any *arrangement* of these elements. Two permutations are alike if, and only if, they have the same elements in the same order. Thus, eat, tea, and ate are the same combination of three letters a, e, t; but they are different permutations of these three letters.

203. **Number of Permutations.** The number of permutations of three elements taken all at a time is 6, as may be seen by writing them down and counting them:

abc, acb, bac, bca, cab, cba.

The number of permutations of 4 elements taken 2 at a time is 12. Thus,

ab, ac, ad; ba, bc, bd;
ca, cb, cd; da, db, dc.

If the number of elements is large the process of counting is tedious. It is possible to derive general formulas for the number of permutations of any number of elements by which the number can be easily computed.

204. **Permutations of *n* Things.** A rule for the number of permutations of n things taken all at a time is easily deduced by means of the principle of § 201. We have n elements and n places to fill. We may think of a row of cells numbered from 1 to n.

1	2	3	4		n

The first cell can be filled in n different ways and after it has been filled the second cell can be filled in $n - 1$ ways. Therefore the first two can be filled in $n(n - 1)$ ways. When they have been filled in any one of these possible ways the third cell can be filled in $(n - 2)$ ways. Therefore the first three cells can be filled in $n(n - 1)(n - 2)$ ways. Continuing thus we see that the first k cells ($k \leqq n$) can be filled in $n(n - 1)(n - 2) \cdots (n - k + 1)$ ways, and that all the n cells can be filled in $n(n - 1)(n - 2) \cdots 2 \cdot 1$ ways. This product of all the natural numbers from 1 to n is called ***factorial*** n, and is denoted by $n!$ or $\lfloor\underline{n}$. Thus, $2! = 2$, $3! = 6$, $4! = 24$, $10! = 3{,}628{,}800$. Therefore,

The number of permutations of n things taken all at a time is factorial n.

For example, 4 horses can be hitched up in 24 ways; 10 cows can be put into 10 stanchions in 3,628,800 ways.

By the same reasoning the number of permutations of n things k at a time ($k \leqq n$) is the number of ways that k cells can be filled from n things. The symbol ${}_nP_k$ is used to denote this number. Then, as shown above,

$$(1) \qquad {}_nP_k = n(n - 1)(n - 2) \cdots (n - k + 1)$$

To remember this formula, note that the first factor is n and the number of factors is k. Thus ${}_5P_3 = 5 \cdot 4 \cdot 3 = 60$. The number of ways in which 4 stanchions can be filled out of a herd of 10 cows is ${}_{10}P_4 = 10 \cdot 9 \cdot 8 \cdot 7 = 5040$. In this notation we should write for the number of permutations of n things all at a time

$$(2) \qquad {}_nP_n = n!$$

205. **Repeated Elements.** The above reasoning assumes that the elements are all distinct. If some of the n elements are alike,

the number of distinguishable permutations is less than $n!$ For example, the number of distinct permutations that can be made out of the 7 letters of the word *reserve* is not 7! The number of permutations of the 7 characters r_1, e_1, s, e_2, r_2, v, e_3 is indeed 7!; but when the subscripts are dropped the permutations $r_1\ e_1\ s\ e_2\ r_2\ v\ e_3$ and $r_2\ e_2\ s\ e_3\ r_1\ v\ e_1$ become identical.

Let x be the number of different permutations of the letters of the word *reserve*. For each of these x there will be 2! permutations of the characters $r_1\ e\ s\ e\ r_2\ v\ e$ and for each of these $x \cdot 2!$ there will be 3! permutations of the characters $r_1\ e_1$-$s\ e_2\ r_2\ v\ e_3$, making $x \cdot 2!\ 3!$ in all. It follows that

$$x \cdot 2!\ 3! = 7! \text{ and } x = \frac{7!}{2!\,3!}$$

This reasoning can be extended to show that *the number of distinguishable permutations of n elements of which p are alike, q others are alike,* etc., $\cdots$, *r others are alike, is equal to*

$$\frac{n!}{p!\ q! \cdots r!} \tag{3}$$

EXERCISES

1. How many 3-letter words can be formed from the letters *a*, *p*, *t*? How many 2-letter words? How many of each are used in the English language?

2. How many different 2-digit numbers can be made from the ten digits 0, 1, 2, $\cdots$, 9? How many if repetitions are allowed? How many of these are used?

3. Find the number of permutations of the letters in each of the following words: (*a*) *degree*, (*b*) *natural*, (*c*) *Indiana*, (*d*) *Mississippi*, (*e*) *Connecticut*, (*f*) *Kansas*, (*g*) *Pennsylvania*, (*h*) *Philadelphia*, (*i*) *Onondaga*, (*j*) *Cincinnati*.

4. In how many ways can a pack of 52 cards be dealt into four piles of 13 each?

5. With 15 players available, in how many ways can the coach fill the various positions on a baseball team?

6. How many different signals of two flags, each one above the other, can be made with five different colored flags?

7. How many different sounds can be made by plucking the five strings of a banjo one or more at a time?

8. How many football signals can be given with four numbers, no repetitions being allowed?

9. In how many ways can four fields be cropped with corn, oats, wheat, and clover, one field to each?

10. A seed store offers 12 varieties of garden seeds. My garden has 8 rows. In how many ways can I plant one row of each variety selected?

11. In how many ways can a gardener plant 2 rows of lettuce, 3 of onions, 3 of beans, 4 of potatoes, if his garden has 12 rows?

12. How large a vocabulary could be formed with 9 letters, no repetitions being allowed? How many with ten? How many with twenty-six? (There are about 100,000 words in Webster's dictionary. The average man has a vocabulary of less than 5000 words.)

206. **Combination of n Things k at a Time.** The symbol ${}_nC_k$ or $\binom{n}{k}$ is used to denote the number of different combinations (§ 202) that can be made from n elements taken k at a time.

A combination of k elements can be arranged into $k!$ permutations of these elements. That is, there are $k!$ times as many permutations as there are combinations of k elements taken all at a time. Whence

$$ {}_nP_k = k!\,{}_nC_k. $$

Making use of the value of ${}_nP_k$, (1), § 203, and solving for ${}_nC_k$ we have,

$$ (4) \qquad {}_nC_k = \frac{n(n-1)(n-2)\cdots(n-k+1)}{1\cdot 2\cdot 3\cdots k}. $$

To remember this formula note that the first factor of the numerator is n, and that there are k factors in the numerator and k in the denominator.

Another useful form of this result is obtained by multiplying

both numerator and denominator of (4) by $(n-k)(n-k-1)(n-k-2)\cdots 2\cdot 1$. This gives

$$(5) \qquad {}_nC_k = \frac{n!}{k!(n-k)!}.$$

We note that the interchange of k and $n-k$ leaves (5) unaltered and hence conclude that

$$(6) \qquad {}_nC_{n-k} = {}_nC_k.$$

This is what we should expect when we think that the numbers of ways that k things can be selected from a group of n must be the same as the number of ways that $n-k$ can be rejected.

EXERCISES

1. From a pack of 52 cards how many different hands can be dealt?
2. How many combinations of 5 can be drawn from 42 dominoes?
3. How many different tennis teams can be made up from 6 players (*a*) singles; (*b*) doubles?
4. How many straight lines can be drawn through 8 points, no three of which lie on a straight line? How many circles?
5. How many diagonals has a convex polygon of n vertices? *Ans.* ${}_nC_2 - n$.
6. Two varieties of corn are planted near each other. How many varieties will be harvested? *Ans.* ${}_2C_2 + 2$.
7. If four varieties of oats are sown near each other, how many varieties will be harvested? *Ans.* ${}_4C_2 + 4$.
8. A starts with two kinds of pure-bred chickens. How many kinds will he have at the end of the third hatching if all stock is sold when one year old? *Ans.* ${}_6C_2 + 6$.
9. In how many ways can 15 gifts be made to 3 persons, 5 to each? *Ans.* ${}_{15}C_5 \cdot {}_{10}C_5$.
10. In how many ways can 15 gifts be made to 3 persons, 4 to A, 5 to B, 6 to C? *Ans.* 630,630.
11. Given (*a*) ${}_nC_2 = 45$; (*b*) ${}_nC_2 = 190$; (*c*) ${}_nC_2 = 105$; find n.
12. In how many different ways can 500 ears of corn be selected from 505 ears?
13. Compute: (*a*) ${}_{1000}C_{997}$; (*b*) ${}_{500}C_{498}$; (*c*) ${}_{10002}C_{10000}$.

CHAPTER XVI

THE BINOMIAL EXPANSION—LAWS OF HEREDITY

207. Product of n Binomial Factors. If the indicated multiplications are performed and terms containing like powers of x are collected,

$$(1)\quad (x + a_1)(x + a_2)(x + a_3)(x + a_4) \cdots (x + a_n) = x^n + C_1x^{n-1} + C_2x^{n-2} + C_3x^{n-3} + \cdots + C_{n-1}x + C_n$$

in which the coefficients have the following values:

$$C_1 = a_1 + a_2 + a_3 + \cdots + a_n.$$

The number of these terms is n.

$$C_2 = a_1a_2 + \cdots a_1a_n + a_2a_3 + \cdots + a_3a_4 + \cdots + a_{n-1}a_n.$$

The number of these terms is the number of combinations that can be made from n a's, 2 at a time, i. e., ${}_nC_2$.

$$C_3 = a_1a_2a_3 + a_1a_2a_4 + \cdots + a_2a_3a_4 + \cdots + a_{n-2}a_{n-1}a_n.$$

The number of these terms is the number of combinations that can be made from n a's, 3 at a time, i. e., ${}_nC_3$.

$$C_4 = a_1a_2a_3a_4 + a_1a_2a_3a_5 + \cdots + a_{n-3}a_{n-2}a_{n-1}a_n.$$

The number of these terms is ${}_nC_4$.

. . . . . .

$$C_r = a_1a_2a_3 \cdots a_r + \cdots.$$

The number of these terms is ${}_nC_r$.

. . . . . .

$C_n = a_1a_2a_3 \cdots a_n$, and consists of one term.

If now each of the a's be replaced by y, it is evident that,

$$C_1 = ny, \qquad C_2 = {}_nC_2y^2, \qquad C_3 = {}_nC_3y^3, \qquad \cdots,$$
$$C_r = {}_nC_ry^r, \qquad \cdots, \qquad C_n = y^n,$$

and therefore

$$(2) \qquad (x+y)^n = x^n + nx^{n-1}y + {}_nC_2x^{n-2}y^2 + {}_nC_3x^{n-3}y^3 + \cdots + {}_nC_rx^{n-r}y^r + \cdots + nxy^{n-1} + y^n.$$

This is known as the binomial expansion, or binomial formula.

208. Binomial Theorem. If x and y are any real (or imaginary) numbers and if n is a positive integer, then the binomial formula (2) is valid. The following observations will be of value.

(1) The exponent of x in the first term is 1 and decreases by 1 in each succeeding term.

(2) The exponent of y in the second term is 1 and increases by 1 in each succeeding term.

(3) The coefficient of the first term is 1, that of the second term is n. The coefficient of any term can be found from the next preceding term by *multiplying the coefficient by the exponent of x and dividing by one more than the exponent of y.*

(4) The $(r+1)$th term is ${}_nC_rx^{n-r}y^r$, i. e.,

$$\frac{n(n-1)(n-2)\cdots(n-r+1)}{r!}x^{n-r}y^r.$$

The coefficient of this $(r+1)$th term is the product of *the first r factors of factorial n, divided by factorial r.*

(5) The sum of the exponents of x and y in any term is n.

(6) The number of terms is $n+1$.

To prove the rule in statement (3) apply it to the $(r+1)$th term,

$${}_nC_r \cdot x^{n-r}y^r.$$

It gives

$$ {}_nC_r \cdot \frac{n-r}{r+1} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!} \cdot \frac{n-r}{r+1} $$
$$ = \frac{n(n-1)(n-2)\cdots(n-r)}{(r+1)!} $$

but this is precisely ${}_nC_{r+1}$, which was to be proved.

209. Binomial Coefficients. The coefficients in the binomial expansion are called binomial coefficients. Their values are given in the following table for a few values of n. This table is called ***Pascal's triangle.***

TABLE OF BINOMIAL COEFFICIENTS, ${}_nC_r$.—PASCAL'S TRIANGLE

	$r=0$	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$	$r=7$	$r=8$	$r=9$	$r=10$	$r=11$
$n=1$	1	1										
$n=2$	1	2	1									
$n=3$	1	3	3	1								
$n=4$	1	4	6	4	1							
$n=5$	1	5	10	10	5	1						
$n=6$	1	6	15	20	15	6	1					
$n=7$	1	7	21	35	35	21	7	1				
$n=8$	1	8	28	56	70	56	28	8	1			
$n=9$	1	9	36	84	126	126	84	36	9	1		
$n=10$	1	10	45	120	210	252	210	120	45	10	1	
$n=11$	1	11	55	165	330	462	462	330	165	55	11	1
etc.	etc.	etc.	....	....	....	....	....	....	....	....	....	....

NOTE. If any number in the table be added to the one on its right, the sum is the number under the latter.

210. Sum of Binomial Coefficients. A great many uses for binomial coefficients and a great many relations among them have been discovered. Two of these are as follows.

(1) *The sum of the binomial coefficients of order n is 2^n.* We verify from the above table that

$$1+1=2^1; \quad 1+2+1=2^2; \quad 1+3+3+1=2^3; \quad \textit{etc.}$$

To prove it for any value of n, put $x=1$ and $y=1$, in the

binomial formula:

$$(1 + 1)^n = 1 + {}_nC_1 + {}_nC_2 + \cdots + {}_nC_{n-1} + {}_nC_n$$

which proves the statement.

Transposing 1, we have

$${}_nC_1 + {}_nC_2 + {}_nC_3 + \cdots + {}_nC_n = 2^n - 1$$

i. e., *the total number of combinations of n things taken* 1, 2, 3, $\cdots$, *n, at a time is* $2^n - 1$.

(2) *The sum of the odd numbered coefficients is equal to the sum of the even numbered ones and each is* 2^{n-1}.

We verify from the table, that

$$1 = 1, \qquad 1 + 1 = 2, \qquad 1 + 3 = 3 + 1,$$
$$1 + 6 + 1 = 4 + 4, \qquad \text{etc.}$$

To prove it for any value of n, put $x = 1$, $y = -1$, in the binomial formula:

$$(1 - 1)^n = 1 - {}_nC_1 + {}_nC_2 - {}_nC_3 + {}_nC_4 - \cdots \pm {}_nC_n$$

whence

$$1 + {}_nC_2 + {}_nC_4 + \cdots = {}_nC_1 + {}_nC_3 + {}_nC_5 + \cdots.$$

211. Use of the Binomial Theorem. In expanding a binomial with a given numerical exponent, the student is urged *to find the successive coefficients by using the statement* (3) § 208, and not by substitution in a formula. This is illustrated in the following examples.

Example 1. Expand $(2x - 3y)^5$.

$$(2x - 3y)^5 = (2x)^5 + 5(2x)^4(-3y)^1 + 10(2x)^3(-3y)^2 + 10(2x)^2(-3y)^3 + 5(2x)(-3y)^4 + (-3y)^5.$$

The coefficients are computed mentally as follows,

the 3d coefficient from the 2d term : $5 \times 4/2 = 10$,
the 4th " " " 3d term : $10 \times 3/3 = 10$,
the 5th " " " 4th term : $10 \times 2/4 = 5$,
the 6th " " " 5th term : $5 \times 1/5 = 1$.

Simplifying the terms, we have

$$(2x - 3y)^5 = 32x^5 - 240x^4y + 720x^3y^2 - 2160x^2y^3 + 810xy^4 - 243y^5.$$

EXAMPLE 2. Expand $(3 - \frac{1}{2})^6$.

$$(3 - \tfrac{1}{2})^6 = 3^6 + 6(3)^5(-\tfrac{1}{2})^1 + 15(3)^4(-\tfrac{1}{2})^2 + 20(3)^3(-\tfrac{1}{2})^3 + 15(3)^2(-\tfrac{1}{2})^4 + 6(3)^1(-\tfrac{1}{2})^5 + (-\tfrac{1}{2})^6.$$

The coefficients are computed as follows:

$6 \times 5/2 = 15$, $15 \times 4/3 = 20$, $20 \times 3/4 = 15$, etc.

Simplifying, we have

+	−
729.	729.
303.75	67.5
8.4375	0.5625
0.015625	797.0625
1041.203125	
797.0625	

$(2\frac{1}{2})^6 = 244.140625$

EXAMPLE 3. Expand $(a + b + c)^3$.

$$\begin{aligned}[(a + b) + c]^3 &= (a + b)^3 + 3(a + b)^2c + 3(a + b)c^2 + c^3 \\ &= a^3 + 3a^2b + 3ab^2 + b^3 + 3(a^2 + 2ab + b^2)c \\ &\qquad + 3(a + b)c^2 + c^3 \\ &= a^3 + b^3 + c^3 + 3a^2b + 3a^2c + 3b^2c + 3b^2a + 3c^2a \\ &\qquad + 3c^2b + 6abc.\end{aligned}$$

EXERCISES

Expand the following expressions by the binomial theorem.

1. $(x + 3)^5$.
2. $(y - 4)^6$.
3. $(2 - x)^4$.
4. $(2x + 3y)^3$.
5. $(3x - 4y)^3$.
6. $(3a + x^2)^6$.
7. $(x^{\frac{1}{2}} + y^{\frac{1}{2}})^4$.
8. $(x^{-1} + 2ay^{-1})^4$.
9. $(a^{-1} - x^{-2})^4$.

10. $(a^2 - b^2)^8$. 11. $(3a^2b + 2c^3)^5$. 12. $(1 + x)^{10}$.
13. $\left(\frac{x}{2} + \frac{2}{x}\right)^{10}$. 14. $(2x - \frac{1}{2})^9$. 15. $(\frac{1}{2} + \frac{1}{3})^8$.
16. $(4.9)^3$. 17. $(1.01)^5$. 18. $(0.99)^6$.
19. $(1.9)^5$. 20. $(1.02)^4$. 21. $(15/8)^7$.
22. Expand $(1 + \frac{1}{3})^5$ and $(2 - \frac{2}{3})^5$ and check results.
23. Prove that any binomial coefficient, counted from the first, is equal to the same numbered one, counted from the last.

212. Selected Terms. To select a particular term in the expansion of a binomial without computing the preceding terms, we can use the formula for the $(r + 1)$th term, namely,

$$ {}_nC_r x^{n-r} y^r = \frac{\text{the first } r \text{ terms of } n!}{r!} x^{n-r} y^r. $$

EXAMPLE 1. Find the 10th term of $\left(\frac{x}{2} - 2y\right)^{20}$.

Here $r + 1 = 10$, $r = 9$, $n = 20$, and the required term is

$$ \frac{20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12}{9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} \left(\frac{x}{2}\right)^{11} (-2y)^9 = -41990x^{11}y^9. $$

EXAMPLE 2. Find the term in $\left(\sqrt{x} + \frac{1}{x}\right)^{13}$ which contains x^2.

The $(r + 1)$th term is ${}_{13}C_r (x^{1/2})^{13-r} (x^{-1})^r = {}_nC_r \cdot x^{(13-3r)/2}$, whence r must be 3 and the 4th term is required. It is

$$ \frac{13 \cdot 12 \cdot 11}{3 \cdot 2 \cdot 1} (x^{1/2})^{10} (x^{-1})^3 = 286x^2. $$

EXERCISES

1. Find the 4th term of $(4a - b)^{12}$.
2. Find the 11th term of $(2x - y)^{17}$.
3. Find the 6th term of $(x\sqrt{y} + y\sqrt{x})^9$.
4. Find the middle term of $(x + 3y)^8$.
5. Find the term of $\left(\frac{x}{2} + \frac{2}{x}\right)^{10}$ which does not contain x.
6. Find the term of $\left(\frac{x}{y} - \frac{y^2}{x^2}\right)^{12}$ which contains neither x nor y.

213. The Binomial Series. The binomial theorem and the symbols ${}_nC_r$ for the number of combinations of n things taken r at a time, have no meaning except when n and r are positive integers. On the other hand we know that such expressions as

$$(1 + \tfrac{1}{2})^{3/2}, \quad (2 + 5)^{-2}, \quad (32 + 3)^{1/5}, \quad (1 - 0.1)^{-1/2},$$

have perfectly definite meanings; e. g., $(2 + 5)^{-2} = 1/49$.

If we should expand a binomial whose exponent is not a positive integer by the binomial theorem (that is form the coefficients and exponents by the same rules as though the exponent were a positive integer), we should get a non-terminating series of terms. For example,

$$(32 + 3)^{1/5} = 32^{1/5} + \tfrac{1}{5}(32)^{-4/5}(3)^1 - \tfrac{2}{2\,5}(32)^{-9/5}(3)^2 + \tfrac{3}{1\,2\,5}(32)^{-14/5}(3)^3 - \cdots.$$

Now it is shown in advanced courses in mathematics, that this binomial series is actually valid, *provided the numerical value of the first term of the binomial is greater than the numerical value of the second term.* It is then valid, in the sense that if we begin at the first and add term after term, the more terms we take the nearer the sum approaches to the true value sought and that, by taking terms enough, the sum which we are computing will approximate the true value as nearly as we please.

Example. Find $\sqrt[3]{10}$ by the binomial series.

$$\sqrt[3]{10} = (8 + 2)^{1/3} = 2(1 + \tfrac{1}{4})^{1/3}$$

$$= 2[1^{1/3} + \tfrac{1}{3}(1)^{-2/3}(\tfrac{1}{4})^1 - \tfrac{1}{9}(1)^{-5/3}(\tfrac{1}{4})^2 + \tfrac{5}{8\,1}(1)^{-8/3}(\tfrac{1}{4})^3 - \tfrac{10}{2\,4\,3}(1)^{-11/3}(\tfrac{1}{4})^4 + \tfrac{22}{7\,2\,9}(1)^{-14/3}(\tfrac{1}{4})^5 \cdots].$$

Whence computing, we have

+	−	
1.0000···	0.0069···	1.0843···
.0833···	.0002···	.0071···
.0010···	.0000···	1.0772···
.0000···		2
1.0843···	0.0071	2.1544··· $= \sqrt[3]{10}$.

The student should note carefully that while the binomial series for $(1 + \frac{1}{4})^{1/3}$ is valid, that for $(\frac{1}{4} + 1)^{1/3}$ is not.

EXERCISES

Expand the following in binomial series and simplify five terms.

1. $(1 + x)^{1/2}$.
2. $(1 + x)^{-1/2}$.
3. $(1 - x)^{-1/3}$.
4. $(0.98)^{1/3}$.
5. $(1.02)^{1/2}$.
6. $(0.99)^{1/2}$.
7. $\sqrt{96}$.
8. $\sqrt[3]{30}$.
9. $\sqrt[4]{60}$.
10. $\sqrt[5]{33}$.
11. $\sqrt[4]{15}$.
12. $\sqrt[3]{65}$.
13. $\sqrt[3]{732}$.
14. $\sqrt[10]{1025}$.
15. $\sqrt[4]{2400}$.
16. $\sqrt[7]{125}$.
17. Show that $\dfrac{1}{\sqrt{1 - x^2}} = 1 + \frac{1}{2}x^2 + \dfrac{1 \cdot 3}{2 \cdot 4}x^4 + \dfrac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^6 + \cdots$.
18. Show that $\dfrac{1}{\sqrt[3]{1 - x}} = 1 + \frac{1}{3}x + \dfrac{1 \cdot 4}{3 \cdot 6}x^2 + \dfrac{1 \cdot 4 \cdot 7}{3 \cdot 6 \cdot 9}x^3 + \cdots$.
19. Show that $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$.
20. Show that $\dfrac{1}{\sqrt{1 + x}} = 1 - \frac{1}{2}x + \dfrac{1 \cdot 3}{2 \cdot 4}x^2 - \dfrac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}x^3 + \cdots$.

214. **Mendel's Law.*** An Austrian monk by the name of Mendel planted some sweet peas of different colors in the garden of the monastery. These blossomed and produced seed. This seed was gathered and planted the following year. The flowers produced the second summer contained all of the colors of the first summer, but other colors were present. By observing and counting the number of flowers of each color Mendel discovered the law which bears his name. In its simplest form it may be explained as follows.

Suppose a bed of sweet peas with blossoms half of which are red and half of which are white. Fertilization of the flowers by wind and insects will take place without selection. That is, pollen from a white flower is equally likely to fertilize a red or a white flower. If pollen from a white flower fertilizes a white

* The following articles (§§ 214–216) are based largely upon Chapter XIV of E. Davenport, *Principles of Breeding*, Ginn and Co. Much additional information may be found there.

flower the seed produced is of *pure* stock and will produce pure white flowers the following year. Such flowers let us denote by W^2. If pollen from a white flower fertilizes a red flower, or vice versa, the seed produced will be *mixed* stock and the following year will show its mixed character by producing flowers which are neither red nor white but some intermediate shade. Such flowers let us denote by RW. The symbol R^2 is now self-explanatory. On counting the flowers which are pure white, mixed, and red, we would discover their numbers to be approximately in the ratio 1 : 2 : 1. These are the coefficients in the expansion of $(R + W)^2$. This is what one might have expected beforehand, as is seen from the adjoined table. Observe that there are twice as many flowers of mixed color as of either of the pure colors.

Color of fertilizing flower.	Color of flower fertilized.	
	R	W
R.....	R^2	RW
W.....	RW	W^2

Result of mixing: $R^2 + 2RW + W^2$.

215. **Successive Generations.** Let R^4 denote the result of fertilizing R^2 with R^2; R^3W denote the result of fertilizing R^2 with RW, and so on. Then the results of indiscriminate fertilization of the flowers will be shown in the second generation, but in the third year, as given in the following table.

Color of fertilizing flower and its relative numbers.	Color of flower fertilized and their relative numbers.		
	R^2	$2RW$	W^2
R^2.....	R^4	$2R^3W$	R^2W^2
$2RW$.....	$2R^3W$	$4R^2W^2$	$2RW^3$
W^2.....	R^2W^2	$2RW^3$	W^4

Result of mixing: $R^4 + 4R^3W + 6R^2W^2 + 4RW^3 + W^4$

Observe that the result in the second generation of mixing is the binomial expansion of $(R + W)^4$.

Similarly we can show that the result in the third generation of mixing is given by $(R + W)^8$, and so on.

216. **Mixing of Three Colors.** Make a table, as above, but for three colors. Suppose the third color to be blue (B). Then a complete expression for the effect, in the first generation after mixing, is the following:

$$(R + W + B)^2 = R^2 + W^2 + B^2 + 2RW + 2RB + 2WB.$$

In case the ratio of the number of white flowers to red flowers is as 2 to 3 then the result in the first generation after mixing is as follows:

$$(2W + 3R)^2 = 4W^2 + 12WR + 9R^2.$$

Mendel's law of heredity, as illustrated above by the distribution of color in the successive generations of plants, applies to other transmissible characters in both plants and animals. That this distribution follows the mathematical laws of the binomial formula is due to the fact that each individual plant or animal inherits the characteristics of *two* parents, and hence the number two and its mathematical properties have their analogies in the laws of biology.

EXERCISES

1. Plot a few graphs, using binomial coefficients as ordinates and the number of the corresponding term as abscissas.

2. How many varieties of sweet peas are produced by sowing in the same bed three different strains (*a*) first year; (*b*) second year.

Ans. (*a*) 6; (*b*) 14.

3. A farmer buys two different kinds of thoroughbred chickens but allows them to mix freely. How many different kinds of chickens will he have at the end of (*a*) the first, (*b*) the second, (*c*) the third year of hatching? *Ans.* (*a*) 3, (*b*) 5, (*c*) 9.

4. Four different varieties of wheat are planted side by side. How many different varieties will be harvested? *Ans.* 10.

5. Plot graphs as indicated in Ex. 1 for the results of Ex. 3.

6. What varieties and in what proportion are obtained by freely mixing the first and second generations?

7. I plant 8 sweet pea seeds — 4 red, 4 white. Each seed produces 16 flowers — each flower matures 2 seeds which germinate and grow the following season. Find the total number of flowers, the proportion and number of the different kinds of flowers, in the (*a*) first, (*b*) second, and (*c*) third generations.

CHAPTER XVII

THE COMPOUND INTEREST LAW

217. Compound Interest. Suppose one dollar to be loaned at compound interest at $r\%$ per annum payable annually. The interest i, due at the end of the first year, is $r/100$. The amount due is $1 + i$. If interest is payable semiannually the amount due at the end of the first half year is $1 + i/2$.* If the interest is payable quarterly the amount due at the end of the first quarter is $1 + i/4$.

In general terms if the interest is payable p times a year at $r\%$ per annum compound, the amounts due on a principal of one dollar at the end of the 1st, 2d, $\cdots$, pth period are respectively,

$$\left(1 + \frac{i}{p}\right), \left(1 + \frac{i}{p}\right)^2, \cdots, \left(1 + \frac{i}{p}\right)^p,$$

and the amounts due at the end of the 1st, 2d, $\cdots$, nth years are respectively,

$$\left(1 + \frac{i}{p}\right)^p, \left(1 + \frac{i}{p}\right)^{2p}, \cdots, \left(1 + \frac{i}{p}\right)^{np}.$$

The amount A at the end of n years at $r\%$ per annum payable p times a year on a principal of P dollars is given by the formula

* The amount of one dollar for n years compound interest at $r\%$ payable annually is $(1 + i)^n$. If a settlement is made between two interest dates there is some divergence of practice in computing the interest for the fractional part of a year. The amount of one dollar for the pth part of a year by analogy to $(1 + i)^n$ would be $(1+i)^{1/p} = \sqrt[p]{1+i}$, but $1 + \frac{i}{p}$ is often used instead. When, however, by the terms of the note the interest is payable p times a year, and is to be compounded, it is clear that the amounts due at the end of $1, 2, \cdots, n$ periods are

$$1 + \frac{i}{p}, \left(1 + \frac{i}{p}\right)^2, \cdots, \left(1 + \frac{i}{p}\right)^n.$$

$$A = P\left(1 + \frac{i}{p}\right)^{np}. \tag{1}$$

218. Continuous Compounding. The larger p is the shorter the interval between the successive interest paying dates. As p increases without bound this interval approaches zero; *i.e.* we can take p large enough to make this interval as small as we please. In the limit interest is said to be compounded continuously. While this state is never realized in financial affairs it is closely approximated. For example, large retail stores sell goods over the counter very nearly continuously and continuously replenish their stock.

Let us see what form equation (1) takes when p becomes infinite. Put x for i/p which approaches zero when p becomes infinite. Then (1) becomes

$$A = P(1 + x)^{\frac{in}{x}} = P[(1 + x)^{\frac{1}{x}}]^{in}. \tag{2}$$

Now it is shown in books on the Calculus that as x approaches zero, the quantity $(1 + x)^{\frac{1}{x}}$ converges to a certain number between 2 and 3. This number is the base of the natural or Napierian system of logarithms and is usually denoted by e. To five decimal places $e = 2.71828$. It can be shown that the following steps are justifiable, although the proof will not be given here. By the Binomial Formula,

$$(1 + x)^{\frac{1}{x}} = 1 + \frac{1}{x}x + \frac{\frac{1}{x}\left(\frac{1}{x} - 1\right)}{2!}x^2 + \frac{\frac{1}{x}\left(\frac{1}{x} - 1\right)\left(\frac{1}{x} - 2\right)}{3!}x^3 + \cdots$$

$$= 1 + 1 + \frac{1 - x}{2!} + \frac{(1 - x)(1 - 2x)}{3!} + \cdots.$$

As x approaches zero the terms on the right converge respectively to the terms of the series

$$1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots.$$

If we begin at the first and add the terms of this series, the more terms we add the nearer the sum comes to e. The sum of the first ten terms is 2.71828, as is shown in the adjoining computation.

2.00000	0
0.50000	0
0.16666	7
0.04166	7
0.00833	3
0.00138	9
0.00019	9
0.00002	5
0.00000	2
2.71828	

Then we conclude that as x approaches zero $(1 + x)^{\frac{1}{x}}$ converges to e.

Returning now to equation (2) we see that as p becomes infinite and x approaches zero, A converges to Pe^{in}. Hence we say that when interest is compounded continuously, the amount of P dollars at $r\%$ per annum for n years is given by the equation

$$A = Pe^{in}, \tag{3}$$

in which $i = r/100$ is the simple interest on one dollar for one year. This equation is said to represent the ***compound interest law.***

Scientific investigations reveal many examples of quantities whose rate of increase (or decrease) varies as the magnitude of the quantity itself. For example, the number of bacteria in a favorable medium, or the growth of an organic body by cell multiplication; again the rate of decrease in atmospheric pressure in ascending a mountain is proportional to the pressure, and the rate of change in the volume of a gas expanding against resistance varies as the volume. The proverbial phrases, *the rich grow richer, the poor poorer; nothing succeeds like success; a stitch in time saves nine;* are expressions in popular language which show a recognition of this law in crude form.*

In general terms if y and x are two varying quantities such that the rate of change in y (as regards a change in x) is known to vary directly as y itself, then they are connected by an equation of the form

* See Davis, *The Calculus*, § 81.

(4) $$y = ce^{kx}$$

in which c and k are constants.

EXAMPLE. Suppose that atmospheric pressure at the earth's surface is 15 lbs. per square inch and that it is 10 lbs. per square inch at a height of 12,000 ft. If now it be assumed that the rate of decrease in the pressure is proportional to the pressure, we have from equation (4)

$$p = ce^{kh}.$$

Substituting $p = 15$ when $h = 0$, we find $c = 15$; then substituting $p = 10$, $h = 12{,}000$, $c = 15$, we find

$$k = \frac{\log_e(\frac{2}{3})}{12000},$$

and these values of c and k give

$$p = 15e^{\frac{h}{12000}\log_e \frac{2}{3}} = 15(\tfrac{2}{3})^{\frac{h}{12000}},$$

by means of which the pressure at any height h can be computed.

This example illustrates the method of solving similar problems which fall under the compound interest law. We assume an equation of the form of (4) and determine the constants c and k by substituting in known pairs of values of x and y. Having determined the constants we insert them in the assumed formula which is then in form to give the value of y corresponding to any value whatever of x.

EXERCISES

1. Do you see any relation between the growth of plants, or the increase in population, and the compound interest law? Is the relation exact? What circumstances tend to limit its application?

2. Is there any relation between your ability to acquire knowledge and to think clearly and the compound interest law?

3. The population of the state of Washington was 349,400 in 1890 and in 1900 it was 518,100. Assume the relation $P = ce^T$, where P = population, T = time in years after 1890, and predict the population for 1910.

4. Using the data of Ex. 3, find the average annual rate of increase from 1890 to 1900. Assuming the same average rate to be maintained for the next 10 years, predict the population for 1910.

5. When heated, a metal rod increases in length according to the compound interest law. If a rod is 40 ft. long at 0° C., and 40.8 ft. long at 100° C., find (*a*) its length at 300° C; (*b*) at what temperature its length will be 41 ft. 6 in. *Ans.* (*a*) 42.448; (*b*) 185°.8

6. The rate of increase in the tension of a belt is proportional to the tension as the distance changes from the point where the belt leaves the driven pulley. If the tension = 24 lbs. at the driven pulley, and 32 lbs. ten feet away, what is it six feet away? *Ans.* 28.52

7. Assuming that the rate of increase in the number of bacteria in a given quantity of milk varies as the number present, if there are 10,000 at 6 A.M., 60,000 at 9 A.M., how many will there be at 2 P.M.? At 3 P.M.? At 6 P. M.? *Ans.* 2 P.M., 1,188,700.

8. In the process of inversion of raw sugar, the rate of change is proportional to the amount of raw sugar remaining. If after 10 hours 1000 lbs. of raw sugar has been reduced to 800 lbs., how much raw sugar will remain at the end of 24 hours? *Ans.* 586 lbs.

CHAPTER XVIII

PROBABILITY

219. Definition of Probability. *If an event can happen in h ways, and fail in f ways, the total number of ways in which the event can happen and fail is $h + f$. Then $h/(h + f)$ is said to be the probability that the event will happen, and $f/(h + f)$ is said to be the probability that the event will fail.*

For example, suppose we have a box containing 4 red marbles and 5 white ones. Let us determine the chance of drawing a red marble the first time. This event can happen in 4 ways, and fail in 5 ways, while the total number of ways in which the event can happen and fail is nine. Then by the preceding definition the probability of drawing a red marble is 4/9, and the probability of not drawing a red marble is 5/9. Observe that one of these things is certain to happen. *The measure of this certainty is the sum of the probabilities of the separate events. This sum is* 1. *Hence, if p is the probability that an event will happen, the probability q that it will not happen is* $1 - p$.

220. Statistical Probability. In a throw of a penny, before the event takes place, there is no reason to suspect that heads are more likely to turn up than tails. In a throw of a die any one of the six faces is equally likely to turn up and this probability does not depend upon the particular die used. The probability of a man's making a safe hit in a game of baseball, and that of not making a safe hit are not equal. Here the individuality of the batter enters and before the event takes place, if the batter

is unknown, we have nothing on which to make an estimate. If the batter is known, our estimate is based on his past performance and this, unlike a throw of dice, depends upon the particular individual at bat. If out of the last 60 times at bat, he has made a safe hit 20 times, then we say that the probability of his making a safe hit this time at bat is 1/3.

Again what is the probability that a man aged 70 will die within the next year? Clearly this depends upon the individual, his present state of health, his habits, etc. In this case, however, we can construct a measure of his probability of dying which is independent of these personal elements. From the *American Experience Mortality Table* (see Tables, p. 329), we find that out of 38,569 persons living at age 70, within the year 2,391 die. Hence the probability that a man aged 70 will die within the year is 2,391 ÷ 38,569.

To derive the probability of an event from statistical data divide the number of cases h in which the event happened by the total number n of cases observed.

221. **Expectation.** If p is the probability that a man will win a certain sum s of money, then the product sp is called the value of his ***expectation.***

Thus the value of a lottery ticket in which the prize is $25 and in which there are 500 tickets is $25 × 1/500, or 30 cents.

EXERCISES

1. According to the mortality table (p. 329) it appears that of 100,000 persons at the age of 10, only 5,485 reach the age of 85. What is the probability that a child aged 10 will reach the age of 85?

2. On 200 of 240 school days a student has had a grade of 90. What is the probability that his grade will be 90 on the 241st day?

3. The weather bureau predicts rain for to-day. What is the probability that it will rain, if on the average 90 out of every 100 predictions are correct?

4. Compute the probability of throwing with 2 dice a sum of (*a*) seven, (*b*) eight, (*c*) nine, (*d*) ten, (*e*) eleven.

Ans. (*a*) $\frac{1}{6}$; (*b*) $\frac{5}{36}$; (*c*) $\frac{1}{9}$; (*d*) $\frac{1}{12}$; (*e*) $\frac{1}{18}$.

5. Find the probability in drawing a card from a pack that it be (*a*) an ace, (*b*) a spade, (*c*) a face card, (*d*) not a face card.

Ans. (*a*) $\frac{1}{13}$; (*b*) $\frac{1}{4}$; (*c*) $\frac{4}{13}$; (*d*) $\frac{9}{13}$.

6. Find the expectation of a man who is to win \$300 if he holds one ticket out of a total of 1000 tickets. *Ans.* 30 cents.

222. **Mutually Exclusive Events.** Two events are said to be ***mutually exclusive*** if the occurrence of one of them precludes the occurrence of the other. For example, in a race between A, B, and C, if A wins, B and C do not win.

If the probabilities of the mutually exclusive events E_1, E_2, $\cdots$, E_n are p_1, p_2, $\cdots$, p_n, *then the probability that some one will occur is the sum of the probabilities of the separate events.*

The meaning will be made clear by means of the following illustration. A bag contains 3 red, 4 white, and 5 blue balls. What is the probability that in a first draw we obtain a red or a white ball? There are 12 balls in all and 7 cases are favorable, namely 3 red and 4 white balls. Then from the definition of probability the chance of drawing a red ball or a white ball is 7/12. But the probability of drawing a red ball is 3/12 and that of drawing a white ball is 4/12 and (3/12) + (4/12) = 7/12.

223. **Dependent Events.** Events are said to be ***dependent*** if the occurrence of one influences the occurrence of the other. *If the probability of a first event is* p_1; *and if after this has happened the probability of a second event is* p_2; *etc.*, $\cdots$; *and if after all those have happened the probability of an nth event is* p_n; *then the probability that all of the events will happen in the given order is* p_1, $p_2 \cdots p_n$.

For, if the first event can happen in h_1 ways and can fail in f_1 ways; and if after this has happened the second can happen in h_2 ways and can fail in f_2 ways; etc., $\cdots$; and if after these have hap-

pened the nth event can happen in h_n ways and can fail in f_n ways; then they can all happen and fail in $(h_1 + f_1)(h_2 + f_2) \cdots (h_n + f_n)$ ways. Now all the events can happen together in the given order in $h_1\, h_2 \cdots h_n$ ways. Then by the definition of probability the chance that all of the dependent events will take place in the given order is

$$\frac{h_1\, h_2 \cdots h_n}{(h_1 + f_1)(h_2 + f_2) \cdots (h_n + f_n)} = \frac{h_1}{h_1 + f_1} \cdot \frac{h_2}{h_2 + f_2} \cdots \frac{h^n}{h_n + f_n}$$

$$= p_1\, p_2 \cdots p_n.$$

Thus the problem of drawing 2 red balls in succession from a bag containing 3 red and 2 black balls is $(3/5) \times (2/4) = 3/10$. For after drawing one red ball and not replacing it the probability of drawing a red ball the second time is 2/4.

224. **Independent Events.** Events are said to be ***independent*** when the occurrence of any one of them has nothing to do with the occurrence of the others.

The probability that all of a set of independent events will take place is the product of the probabilities of the independent simple events.

This follows as a corollary from the theorem of § 223.

Thus the probability of throwing a deuce twice in succession is $(1/6) \times (1/6) = 1/36$.

EXERCISES

1. If the batting average of Tyrus Cobb is 0.400 what is the chance that in any single time at bat he will make a safe hit?

2. What is the probability of holding 4 aces in a game of whist?

Ans. 1/270,725.

3. Suppose I enter 2 horses for a race and that the probabilities of their winning are respectively $\frac{1}{2}$ and $\frac{1}{4}$. What is the probability that one or the other will win the race? *Ans.* 3/4.

4. Does Ex. 3 teach us anything with respect to diversified farming? Discuss the probability of crop failure of a single crop as compared with that of two or more different crops.

5. Three men A, B, C go duck hunting. A has a record of one bird

out of two, B gets two out of three, C gets three out of four. What is the probability that they kill a duck at which all shoot at once?

Ans. 23/24.

6. What is the chance of drawing a white and red ball in the order named from a bag containing 5 white and 6 red balls? *Ans.* 3/11.

7. In a certain zone in times of war 23 out of 5000 ships are sunk by submarine in one week. What is the chance that a single vessel will cross the zone safely? What is the chance that all of 4 vessels which enter the zone at the same time will cross in safety? What is the chance that of these 4 exactly 3 will cross in safety? That at least 3 will cross in safety?

8. In certain branches of the army service 2% of the men are killed each year. Three brothers enlist in this branch of the service for a period of two years. Compute the probability that (*a*) all will survive, (*b*) exactly 2 will survive, (*c*) at least 2 will survive, (*d*) exactly one will survive, (*e*) at least one will survive, (*f*) none will survive.

9. At the time of marriage the probabilities that a husband and wife will each live 50 years are $\frac{1}{5}$ and $\frac{1}{4}$ respectively. Compute the probability that (*a*) both will be alive, (*b*) both dead, (*c*) husband alive and wife dead, (*d*) wife alive husband dead.

10. From the American Experience Table of Mortality (Tables, p. 329) compute your chances of living 1, 10, 20, 30, 40, 50 years.

11. From the American Experience Table of Mortality (Tables, p. 329) find that age to which you now have an even chance of living.

12. Find from the same table that age to which a person aged 20 has an even chance of living. *Ans.* 66^{+}.

13. Three horses are entered for a race. The published odds are 5 : 4 for A; 3 : 2 against B; 4 : 3 against C. Is it possible to place bets in such a way that I win some money no matter which horse wins?

Ans. Yes.

14. Suppose n horses entered for a race, and let the published odds be $(a - 1)$ to 1 against the first; $(b - 1)$ to 1 against the second, $(c - 1)$ to 1 against the third and so on. A man bets $(a - 1)/a$ to $1/a$ against the first; $(b - 1)/b$ to $1/b$ against the second, etc. Show that whatever horse wins his gains are represented algebraically by the formula

$$\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \cdots\right) - 1.$$

225. **Frequency Distribution Curves.*** A sample of 400 oats plants were taken from an experimental plot and measured as to height in centimeters with the following results: †

Height, H	45–50	50–55	55–60	60–65	65–70	70–75	75–80	80–85	85–90	90–95
Frequencies, F	2	9	21	34	97	123	89	24	0	1

Let us plot this data with heights as abscissas and frequencies as ordinates. Construct rectangles, with bases on the horizontal

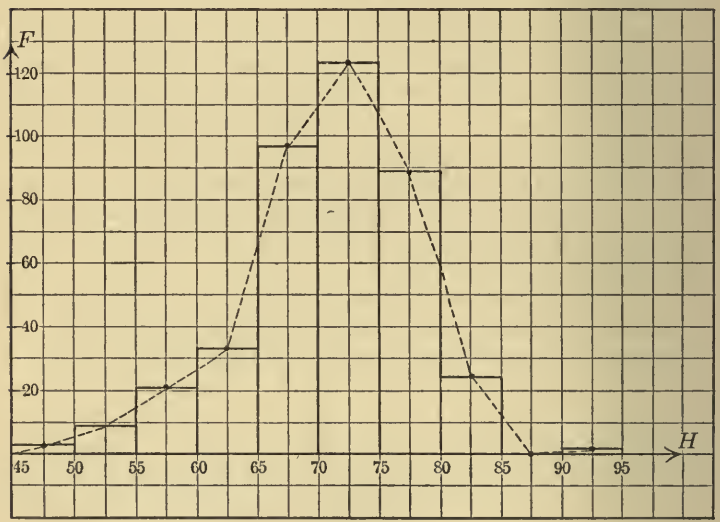


FIG. 128

axis. Let the width of the base in each case be 5 units, which agrees with the grouping of the measurements as to height.

* In the remainder of this Chapter (§§ 225–231), the authors are indebted for many ideas to E. DAVENPORT, *Principles of Breeding* [Chapter XII and Appendix (H. L. RIETZ)]. Other books containing similar matter are JOHNSON, *Theory of Errors and Method of Least Squares*; WRIGHT AND HAYFORD, *Adjustments of Observations*; MERRIMAN, *Textbook of Least Squares*; WELD, *Theory of Errors and Least Squares*; etc.

† MEMOIR NO. 3, CORNELL UNIVERSITY AGRICULTURAL EXPERIMENT STATION, *Variation and Correlation of Oats*, by H. H. LOVE and C. E. LEIGHTY, Aug., 1914.

Let the height of the individual rectangles be representative of the frequency for the corresponding heights of plants, as shown in Fig. 128.

The upper parts of these rectangles form an irregular curve made up of segments of straight lines. A smoother curve is obtained by connecting the middle points of the upper bases of these rectangles by segments of straight lines as shown by the dotted line in Fig. 128. Instead of the dotted line we may draw a smooth curve as near as possible to the middle points of the upper bases. Any curve drawn as nearly as possible through a series of plotted points representing a distribution with respect to a given character is called a ***frequency distribution curve.***

Such curves are useful in presenting to the eye some of the features of a distribution. The type of character most frequent is represented by the ***mode*** (§ 198), which is the value of the abscissa corresponding to the highest point of the curve. The ***median*** measurement of the group (§ 197) is represented by the abscissa of that ordinate on either side of which there are equal areas under the curve. The ***arithmetic*** average (§ 195) is the abscissa of the center of gravity of the area under the curve.

Frequency distribution curves are plotted for a great variety of things, such as frequency distribution of people with respect to height, weight, or age; grains of wheat with respect to weight; alfalfa with respect to duration of bloom in days; cherry trees with respect to earliness of bloom; pigs with respect to size of litter; diphtheria with respect to time of year; women with respect to age of marriage; etc.

226. **Probability Curve.** If a large number of measurements are made upon the same item, they will not in general agree. Let us plot as abscissas the measurements observed and as ordinates their relative frequencies. In most cases, the positive

and negative errors are equally likely to occur, and small errors are more numerous than large ones. The frequency curve for the observed data would then have its highest point at the true value of the measured magnitude, would be symmetric about an ordinate through this highest point, and would rapidly approach the axis of abscissas both to the right and left of this maximum ordinate. If we take the vertical through the highest point as an axis of y, then abscissas will represent errors of observation and ordinates will represent frequency of error.

The curve so drawn is well represented by the equation

$$(1) \qquad y = \frac{n}{\sigma\sqrt{2\pi}} e^{-(x^2/2\sigma^2)}$$

in which σ is what we shall call the ***standard deviation***, $e = 2.71828 \cdots$ the base of Napierian logarithms, n the number of observations, x the error of a reading, y the probability of an

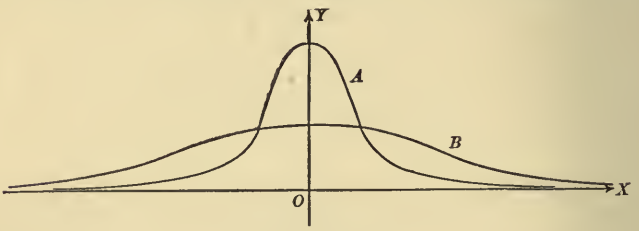


Fig. 129

error x. This curve is called the ***probability curve*** or ***curve of error.***

While the theoretical curve (1) is symmetric, the curves obtained by plotting the results of statistical study are often not symmetric. However the formulas developed in this chapter for the symmetric case can be used for approximate results in the non-symmetric cases.

227. **Standard Deviation.** It is not enough to know the value

of the arithmetic average or the mode. It is important to have a measure of the tendency to deviate from the average or from the mode.

The general theory will be explained by means of the data of § 225, which represents the measurements of the heights of 400 oat plants. From this data the average height of oat plant is 70.8 centimeters. Compute the deviation, D, of these plants from their average height. Multiply the square of each deviation by its corresponding frequency and add the results. We get 19,320. Divide by the sum, 400, of the frequencies. The quotient is 48.3. We next extract the square root since the deviations have all been squared in the above calculations. We get 6.95^-, and this is called the standard deviation.

In general, to find the standard deviation,

Compute the deviation of each frequency from the arithmetic average. Multiply the square of each deviation by its corresponding frequency and add the results. Divide by the sum of the frequencies. Extract the square root.

This rule is symbolized in the following formula:

$$\sigma = \sqrt{\frac{\Sigma D^2 f}{n}}. \tag{2}$$

The curve A in Fig. 129 represents the distribution when σ is small, and the curve B represents the distribution when σ is large.

For example, the two sets of numbers 7, 7, 8, 8, 8, 8, 9, 9 and 5, 6, 7, 8, 8, 9, 10, 11 have the same arithmetic mean. The second set, however, shows a greater tendency to vary from the arithmetic average (type) than does the first. This greater tendency to vary is shown by the larger value for σ for the second set. The values of σ are 0.706 and 1.87 respectively.

Again, suppose two men are shooting at a mark, and that we compute the standard deviation for each. The man for whom σ is smallest is said to be the more consistent shot.

228. **Coefficient of Variability.** A comparison of the standard deviations of two different groups conveys little information as to their respective tendencies to deviate from the arithmetic average. This is due to two causes: (1) the measurements may be in different units, as centimeters and grams, (2) one average may be much larger than the other, for example the average height of a group of men would be larger than the average length of ears of corn. We need then a measure of variability which is independent of the units used and takes into account the relative magnitudes of the means. Such a measure is the ***coefficient of variability***, which is denoted by C and is determined by the formula,

$$C = \frac{\text{Standard deviation}}{\text{Arithmetic average}} = \frac{\sigma}{m}. \tag{3}$$

For example, the coefficient of variability in height of the 400 oat plants considered in § 225 is 6.95/70.8, or approximately 10%.

229. **Probable Error of a Single Measurement.** Any individual measurement is likely to be in error. This error is approximately the difference between this measurement and the arithmetic average of all the measurements. Compute these errors for all the measurements, some positive, some negative. Give them all positive signs and arrange them in order of magnitude. The *median* of this list is called the ***probable error of a single measurement*** of the set and is denoted by E_s. It is shown in the theory of probability that

$$E_s = 0.6745\sigma. \tag{4}$$

230. **Probable Error in the Arithmetic Average.** Take a sample of 500 ears of corn from a crib. Compute the arithmetic average of their lengths. We use this to represent the mean length of all the ears in the crib. Quite likely it differs from their true arithmetic average. We now find by means of equation

(5) below, a number E_m, called the ***probable error in the arithmetic average.*** This is a number such that it is equally likely whether or not the computed arithmetic average of the 500 ears selected lies between $m - E_m$ and $m + E_m$, where m denotes the (unknown) true arithmetic average for all the ears in the crib. In other words if a very large number of persons take a sample of ears and each computes an average length, then, in a sufficiently large number of cases, one half of these averages will be within the limits set and one half will be without.

In treatises on probability it is shown that

$$E_m = \frac{E_s}{\sqrt{n}} = \frac{0.6745\sigma}{\sqrt{n}}. \tag{5}$$

This formula shows that in order to double the precision of the computed arithmetic average it is necessary to take four times as many observations.

231. Probable Error in the Standard Deviation. Compute the standard deviation, § 227, of the lengths of 500 ears of corn from a crib. This will differ slightly from the true standard deviation σ, of the lengths of all the ears in the crib. Next find, by means of equation (6) below, the probable error E_σ, of the standard deviation. Then for a sufficiently large number of samples from the crib, the computed standard deviations will fall one half within the limits $\sigma - E_\sigma$ and $\sigma + E_\sigma$, and one half without. The formula for the probable error in the standard deviation is

$$E_\sigma = \frac{E_m}{\sqrt{2}} = \frac{0.6745\sigma}{\sqrt{2n}}. \tag{6}$$

EXERCISES

1. Compute E_s, E_m, E_σ for the data in § 225.
2. Compute σ, C, E_s, E_m, E_σ for the following sets of measurements.

(*a*) 5, 6, 7, 8, 8, 9, 10, 11; (*b*) 5, 5, 5, 7, 9, 10, 11, 12.

(*c*) 1, 6, 8, 8, 8, 8, 10, 15; (*d*) 51, 56, 58, 58, 58, 58, 60, 65.

3. Compute σ, C, E_s, E_m, E_σ for the following distribution of oat plants with respect to height in centimeters [Love-Leighty].

(*a*)

Height.....	60	65	70	75	80	85	90
Frequency....	2	11	45	140	122	73	7

(*b*)

Height.....	60	65	70	75	80	85	90	95
Frequency....	11	36	60	94	99	102	68	18

4. Compute from the following data the mode, the mean, the coefficient of variability, the standard deviation, the probable error in the mean, and the probable error in the standard deviation.

Lbs. of butter fat..	400	375	350	325	300	275	250	225	200
No. of cows.....	1	2	4	5	7	6	5	2	1

Draw the distribution curve.

5. The following table is taken from Bulletin 110, part 1, *Bureau of Animal Husbandry, U. S. Dept. of Agriculture* on "A Biometrical Study of egg production in the domestic fowl" and shows the frequency distribution for hens in first-year egg production.

Annual Egg Production	0 14	15 29	30 44	45 59	60 74	75 89	90 104	105 119	120 134	135 149	150 164	165 179	180 194	195 209	210 224	225 239
1902–03	...	2	...	1	5	8	17	18	17	26	17	18	9	2	6	1
1903–04	7	5	5	10	10	20	24	29	52	37	29	16	8	2	...	...
1905–06	...	...	1	2	4	9	13	25	24	22	32	17	20	9	...	...
1906–07																
(*a*)	2	2	5	5	9	16	30	39	26	21	19	12	1	...	...	...
(*b*)	10	8	8	15	29	32	48	39	36	25	18	6	5	...	2	...

From this data compute for each year the mean, the median, and the mode for egg production. Compute σ, C, E_σ, E_m, E_s. Draw the distribution curve.

6. From Table I at the end of Chapter XIX compute for each weight (length) the mean, the median, and the mode for length (weight). Compute σ, C, E_σ, E_m, E_s of weight (length) for each length (weight).

7. For Table II (p. 312) follow the directions as given in Ex. 6 for Table I, reading however number of kernels instead of weight.

8. For Table III (p. 312) follow the directions as given in Ex. 6 for Table I, reading yield and number of culms in place of weight and length.

9. For Table IV (p. 313) follow the directions as given in Ex. 6. Read height of mid-parent and height of adult children in place of weight and length.

CHAPTER XIX

CORRELATION*

232. **Meaning of Correlation.** Whenever two quantities are so related that an increase in one of them produces or is accompanied by an increase in the other and the greater the increase in the one the greater the increase in the other, these quantities are said to be ***correlated positively.*** If an increase in one produces, or is accompanied by, a decrease in the other, they are said to be ***correlated negatively.*** If a change in one is not accompanied by any change in the other, there is no correlation, and the quantities are said to be *unrelated.* Perfect positive correlation is represented by the number $+1$, perfect negative correlation by -1, no correlation by zero. There is perfect positive correlation between the area of a rectangular field and its length, the extension of a spiral spring and the suspended load. There is perfect negative correlation between the pressure and volume of a perfect gas. No relation exists between the price of coal and the length of ears of corn.

There are quantities, common in everyday life, such that a change in one is not accompanied by a proportionate change in the other, but a given change in one is always accompanied by some change in the other. Such quantities are still said to be correlated. The degree of relationship may be anywhere between complete independence and complete dependence, that is

* Throughout this Chapter, the authors have consulted the following books, and are indebted to them for ideas: E. Davenport, *Principles of Breeding* (Chap. XIII); Zizek, *Statistical Averages;* Secrist, *Introduction to Statistical Methods;* Pearson, *Grammar of Science;* Bowley, *Elements of Statistics.*

between zero and + 1 or between zero and − 1. For example we may mention the effect of potato prices on acreage, and vice versa.

We desire a numerical measure for this correlation. Any adequate expression must be such that it becomes zero when there is no correlation, − 1 when there is perfect negative correlation, + 1 for perfect positive correlation, and which is always between − 1 and + 1. Yule has proposed a formula which satisfies these conditions. Arrange the observed data with reference to the two quantities in question as in the following diagram:

	x present.	x absent.
y present.....	u	v
y absent.....	r	s

Then a measure m of the correlation existing is given by the equation

$$(1) \qquad m = \frac{us - rv}{us + rv}.$$

If either r or v is zero $\quad m = +1.$

If either u or s is zero $\quad m = -1.$

If $us = rv$ $\quad m = 0.$

EXERCISES

1. Compute from the following table the degree of effectiveness of vaccination against diphtheria:

	Recoveries.	Deaths.
Vaccinated	2843	106
Not vaccinated.....	254	225

2. Compute from the following table the correlation between prohibition and the arrests per day in a given city for one year:

	Days with more than 20 arrests.	Less than 20.
Wet	281	84
Dry	142	223

3. Compute the correlation between use of fertilizer and yield of potatoes in bushels per acre when the results from fifty plats are as follows:

	Yield over 100 bushels.	Under 100 bushels.
Fertilizer	47	3
No fertilizer	14	36

Ans. 0.95

This high value of correlation is considered evidence of some connection between use of fertilizer and yield.

233. Correlation Table. Let it be proposed to find the degree of correlation, if any, between the lengths of ears of corn and their weight, between their lengths and number of rows of kernels, between length and circumference, between length and yield per acre, between length of head of wheat and yield per acre, between height of wheat and yield per acre. The problem is now more complex. Let us take for example a given number of ears of corn and examine them as to weight in ounces and length in inches. The measurements may be tabulated as shown in the accompanying table. Each column is a frequency distribution of lengths for a constant weight. Each row is a frequency distribution of weights for a constant length. The distribution of the ears of length 8 inches with respect to weight is 3, 7, 19, 25, 17, 22, 17, 3, 1.

It is to be noticed that the table extends across the enclosing rectangle from the upper left-hand corner to the lower right-hand corner. Whenever data tabulated with respect to two measurable characters show this skew arrangement, correlation exists. In the accompanying table weights increase from left

to right and lengths increase as we move downward. We have in this case positive correlation. An extension of the array from the upper right-hand corner to the lower left would have indicated negative correlation.

234. **Coefficient of Correlation.** The method of obtaining the correlation coefficient may be explained in connection

CORRELATION BETWEEN WEIGHT AND LENGTH OF EAR *

Length of Ear in Inches.	Weight of Ear in Ounces.																		
	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
3	1	2		1																
3.5		4		1																
4	3	5	5	1																
4.5		6	5	4			1													
5		2	4	7	2	4														
5.5		2	9	15	14	8	4	1												
6		1	2	12	16	13	13	6	1											
6.5			1	6	11	26	11	8	6	1										
7			1	2	2	12	18	12	12	11	4	1								
7.5				1	2	4	20	12	13	21	11	6	6	1	1					
8						3	7	19	25	17	22	17	3	1						
8.5						1	1	12	9	23	30	26	26	5	1					
9								1	7	10	23	35	26	24	12	1	2	1		
9.5									1	4	14	19	29	17	10	1	3		1	1
10									1	1	3	8	18	10	6	4	2			
10.5												2	3	6	7	2	5	1		
11													1	1				2	1	
11.5																	1			

with the above table. Find the arithmetic mean of each character involved — in this case mean length of ear, M_l, and mean weight of ears, M_w. Find the deviation D_l of ear length from mean length, and the deviation D_w of weight from mean weight, for each ear tabulated. For each ear tabulated find the product of D_l and D_w and then add all of these products. This sum we will indicate by $\Sigma D_l D_w$. Find in the usual way the standard deviation of length of ears, σ_l, and the standard deviation of weight of ears, σ_w. Then the coefficient of correlation, r, is

* E. DAVENPORT, *Principles of Breeding*, p. 458.

given by the formula

$$r = \frac{\Sigma D_l D_w}{n\sigma_l\sigma_w}, \tag{2}$$

where n is the number of things observed, in this case the total number of ears.

A convenient arrangement for computing D_l for each ear length and D_w for each ear weight is shown in the table below.

The row labeled 6.5 inches (table § 233), gives the frequency distribution of ears with respect to weight. There is one ear of weight 4 oz., 6 ears of weight 5 oz., 11 ears of weight 6 oz., 26 ears of weight 7 oz., etc.; a total of 70 ears, f_l, of length 6.5 inches.

$$f_l V_l = 1 \times 4 + 6 \times 5 + 11 \times 6 + 26 \times 7 + 11 \times 8 + 8 \times 9 + 6 \times 10 + 1 \times 11 = 455.0$$

The mean length of ear is obtained by adding the numbers in the column headed $f_l V_l$ and dividing this sum by the total number, $n = 993$, of ears.

CORRELATION OF WEIGHT TO LENGTH OF EARS OF CORN

Length, Inches.	f_l	$f_l V_l$	D_l	Weight, Ounces.	f_w	$f_w V_w$	D_w	$D_l D_w$
3	4	12.0	− 4.8	2	4	8	− 8.7	143.0
3.5	5	17.5	− 4.3	3	22	66	− 7.7	156.9
4	14	56.0	− 3.8	4	27	108	− 6.7	394.4
4.5	16	72.0	− 3.3	5	50	250	− 5.7	347.2
5	19	95.0	− 2.8	6	47	282	− 4.7	297.6
5.5	53	291.5	− 2.3	7	71	497	− 3.7	618.9
6	64	384.0	− 1.8	8	75	600	− 2.7	465.8
6.5	70	455.0	− 1.3	9	71	639	− 1.7	306.8
7	75	525.0	− 0.8	10	75	750	− 0.7	110.8
7.5	98	735.0	− 0.3	11	88	968	0.3	14.9
8	114	912.0	0.2	12	107	1,284	1.3	1.4
8.5	134	1,139.0	0.7	13	114	1,482	2.3	129.6
9	142	1,278.0	1.2	14	112	1,568	3.3	466.3
9.5	100	950.0	1.7	15	65	975	4.3	564.4
10	53	530.0	2.2	16	37	592	5.3	431.0
10.5	26	273.0	2.7	17	8	136	6.3	364.0
11	5	55.0	3.2	18	13	234	7.3	107.2
11.5	1	11.5	3.7	19	4	76	8.3	27.0
	993	7,791.5		20	2	40	9.3	
				21	1	21	10.3	
					993	10,576		

$$M_l = \frac{7791.5}{993} = 7.85 \qquad M_w = \frac{10,576}{993} = 10.65$$

* E. DAVENPORT, *Principles of Breeding*, p. 461.

All of the symbols used have been defined with the exception of the following: σ_l is the standard deviation of length; f_w is the number (frequency) of ears of same weight w; V_l stands for the value of length of ears with given frequency; V_w represents the value of weight of ears with given frequency. This gives $M_l = 7.85$. In the row labeled 6.5 and in the column headed D_l we write the difference between this mean length 7.85 and the length 6.5. This gives the number -1.3 of the column headed D_l. The number 306.8 in the last column is obtained as follows:

$$(-1.3)[1(-6.7) + 6(-5.7) + 11(-4.7) + 26(-3.7) + 11(-2.7) + 8(-1.7) + 6(-0.7) + 1(0.3)] = 306.8$$

That is, the ear of weight 4 oz. deviates from the mean weight by 6.7 oz., the 6 ears of weight 5 oz. deviate from the mean weight by 5.7 oz., the 11 ears of weight 6 oz. deviate from the mean weight by 4.7 oz., etc.

The number 306.8 represents the sum of the products of the corresponding length and weight deviations for every individual in the horizontal row to which the number belongs. To find the correlation coefficient add the numbers in the column headed D_lD_w, obtaining in this case 4947.2.

Divide this number 4947.2 by $n \times \sigma_l \times \sigma_w$. In this case $n = 993$, and σ_l, σ_w have been computed to be 1.57 and 3.63 respectively. This gives the correlation coefficient

$$r = \frac{4947.2}{993(1.57)(3.63)} = 0.87$$

235. **The Regression Curve.** For each recorded weight (see table, § 233) compute the arithmetic average of length of ears. Thus the ears of weight 4 oz. have an average length of 5.1 inches. The ears of weight 5 oz. have an average length of 5.46 inches, etc. Plot a curve using for abscissas the weights, and for ordinates the computed average lengths. The curve so plotted is called a ***regression curve.*** In many cases this curve is a straight line. It can be shown that the straight line which best represents the plotted data is given by the equation

(2) $$M_l = r\frac{\sigma_l}{\sigma_w}w.$$

Another regression curve can be plotted for the same data, using lengths as ordinates and mean weights for abscissas. This curve does not in general coincide with the first. Its equation is

$$M_w = r\frac{\sigma_w}{\sigma_l}l.$$

By means of these curves the mean value of one character can be read off when a fixed value is given to the other character.

EXERCISES

1. Find, for the correlation table in § 233:

(*a*) the regression of weight relative to length;

(*b*) regression of length relative to weight.

Ans. (*a*) 2.03 (*b*) 0.38

2. Find the equation of the line of regression in both cases of Ex. 1.

3. Plot the line of regression in Ex. 2 from the equation found there and then again plot the line from the data as suggested in § 235.

4. From Table II, p. 312, which gives the correlation of height of oat plants with the average number per plant of kernels per culm, compute the mean height, the mean number of kernels per culm, the standard deviation with respect to height, the standard deviation with respect to number of kernels per culm, the correlation coefficient, and the regression coefficients.

5. Examine Table IV, p. 313, which gives the number of children of various statures born of 205 mid-parents of various statures. From this table compute:

M_p = mean height of mid-parents,

M_c = mean height of adult children,

σ_p = standard deviation of height of mid-parents,

σ_c = standard deviation of height of adult children,

r = the correlation coefficient, and both regression coefficients.

6. For Ex. 4 plot the lines of regression (*a*) from their equations, (*b*) from the data directly.

7. For Ex. 5 plot the lines of regression (*a*) from their equations, (*b*) from the data directly.

8. From the following table find a measure of the effectiveness of vaccination against smallpox.

	Recoveries.	Deaths.	Total.
Vaccinated	3,951	200	4,151
Not vaccinated	278	274	552
Total	4,229	474	4,703

9. Construct a correlation table from your own observations on length and breadth of leaves. (*a*) Use 30 classes for length. (*b*) Use 15 classes for length, thus making the class interval twice as large. Compute in each case the correlation coefficient.

10. From Table I, below, which gives the correlation of lengths and weights of ears of corn, compute the mean length, the mean weight, the standard deviation with respect to length, the standard deviation with respect to weight, the correlation coefficient, and both regression coefficients.

11. The same as Ex. 10 after writing number of kernels in place of weight, using Table II, p. 312, in place of Table I.

I. Correlation of Length and Weight of Ears of Corn

Length in Inches.	Weight in Ounces.																
	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
3.0		1	1	1													
3.5		1	2	2	1												
4.0	2	3	5	4	1												
4.5		4	5	6	2	1											
5.0		4	7	8	6	4	1										
5.5		3	9	12	13	8	3	1									
6.0		1	5	10	15	12	9	5	2								
6.5			2	6	12	26	14	10	5	3	1						
7.0			1	3	4	14	18	15	10	7	2	1					
7.5				1	2	6	13	17	19	13	9	6	4	2			
8.0						2	7	10	13	19	7	6	2	1			
8.5							1	3	9	14	25	17	8	5	1		
9.0								1	4	7	19	25	16	11	3		
9.5									2	3	8	18	20	15	6	1	
10.0										1	3	9	18	13	7	5	2
10.5												2	3	7	5	4	1
11.0													1	2	3	2	

II. Correlation of Average Height of Oat Plants in Centimeters and Average Number of Kernels per Culm per Plant. [Love-Leighty.] $r = 0.73$.

Height.	Number of Kernels.									
	30 40	40 50	50 60	60 70	70 80	80 90	90 100	100 110	110 120	120 130
55–60	1		1							
60–65		4	7							
65–70			7	22	9	6	1			
70–75			1	13	30	59	32	5		
75–80				2	16	40	38	23	3	
80–85					1	12	26	23	9	2
85–90								3	2	2

III. Correlation of Number of Culms per Oat Plant and Total Yield of Plant in Grams. [Love-Leighty.] $r = 0.712$

Yield	Number of Culms per Plant.					
	2	3	4	5	6	7
0–1	3					
1–2	28	19	3			
2–3	18	66	20	1		1
3–4	1	42	58	7	1	
4–5		7	59	11	3	
5–6			26	14	2	
6–7				4	3	
7–8			1	1		
8–9					1	

IV. Correlation of Heights of Adult Children and Parents

Data for Children of 205 Mid-parents* of Various Statures

Heights of Mid-parents.	Heights of Adult Children in Inches.												
	Above.	73.2	72.2	71.2	70.2	69.2	68.2	67.2	66.2	65.2	64.2	63.2	62.2	Below.
Above		3	1											
72.5	4	2	7	2	1	2	1							
71.5	2	2	9	4	10	5	3	4	3	1				
70.5	3	3	4	7	14	18	12	3	1	1		1		1
69.5	5	4	11	20	25	33	20	27	17	4	16	1		
68.5		3	4	18	21	48	34	31	25	16	11	7		1
67.5			4	11	19	38	28	38	36	15	14	5	3	
66.5					4	13	14	17	17	2	5	3	3	
65.5			1	2	5	7	7	11	11	7	5	9		1
64.5								5	5	1	4	4	1	1
Below						1	1	2	2	1	4	2		1

* Height of mid-parent is the mean height of the two parents.
[Galton-Davenport]

GREEK ALPHABET

Letters	Names	Letters	Names	Letters	Names	Letters	Names
Α α	Alpha	Η η	Eta	Ν ν	Nu	Τ τ	Tau
Β β	Beta	Θ θ	Theta	Ξ ξ	Xi	Υ υ	Upsilon
Γ γ	Gamma	Ι ι	Iota	Ο o	Omicron	Φ ϕ	Phi
Δ δ	Delta	Κ κ	Kappa	Π π	Pi	Χ χ	Chi
Ε ϵ	Epsilon	Λ λ	Lambda	Ρ ρ	Rho	Ψ ψ	Psi
Ζ ζ	Zeta	Μ μ	Mu	Σ σs	Sigma	Ω ω	Omega

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Table I. Logarithms of Numbers

N.	0	1	2	3	4	5	6	7	8	9
0	——	0000	3010	4771	6021	6990	7782	8451	9031	9542
1	0000	0414	0792	1139	1461	1761	2041	2304	2553	2788
2	3010	3222	3424	3617	3802	3979	4150	4314	4472	4624
3	4771	4914	5051	5185	5315	5441	5563	5682	5798	5911
4	6021	6128	6232	6335	6435	6532	6628	6721	6812	6902
5	6990	7076	7160	7243	7324	7404	7482	7559	7634	7709
6	7782	7853	7924	7993	8062	8129	8195	8261	8325	8388
7	8451	8513	8573	8633	8692	8751	8808	8865	8921	8976
8	9031	9085	9138	9191	9243	9294	9345	9395	9445	9494
9	9542	9590	9638	9685	9731	9777	9823	9868	9912	9956
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
N.	0	1	2	3	4	5	6	7	8	9

Prop. Parts		
	22	21
1	2.2	2.1
2	4.4	4.2
3	6.6	6.3
4	8.8	8.4
5	11.0	10.5
6	13.2	12.6
7	15.4	14.7
8	17.6	16.8
9	19.8	18.9
	20	19
1	2.0	1.9
2	4.0	3.8
3	6.0	5.7
4	8.0	7.6
5	10.0	9.5
6	12.0	11.4
7	14.0	13.3
8	16.0	15.2
9	18.0	17.1
	18	17
1	1.8	1 7
2	3.6	3.4
3	5.4	5.1
4	7.2	6.8
5	9.0	8.5
6	10.8	10.2
7	12.6	11.9
8	14.4	13.6
9	16.2	15.3
	16	15
1	1.6	1.5
2	3.2	3.0
3	4.8	4.5
4	6.4	6.0
5	8.0	7.5
6	9.6	9.0
7	11.2	10.5
8	12.8	12.0
9	14.4	13.5
	14	13
1	1.4	1.3
2	2.8	2.6
3	4.2	3.9
4	5.6	5.2
5	7.0	6.5
6	8.4	7 8
7	9.8	9.1
8	11.2	10.4
9	12.6	11.7
	12	11
1	1 2	1.1
2	2.4	2.2
3	3.6	3.3
4	4.8	4.4
5	6.0	5.5
6	7.2	6.6
7	8.4	7.7
8	9.6	8.8
9	10.8	9.9
	9	8
1	0.9	0.8
2	1.8	1.6
3	2.7	2.4
4	3.6	3.2
5	4.5	4.0
6	5.4	4.8
7	6.3	5.6
8	7.2	6.4
9	8.1	7.2

Table I. Logarithms of Numbers

N.	0	1	2	3	4	5	6	7	8	9	Prop. Parts
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	9
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	1 0.9
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	2 1.8
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	3 2.7
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	4 3.6
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	5 4.5
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	6 5.4
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	7 6.3
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	8 7.2
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	9 8.1
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	8
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	1 0.8
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	2 1.6
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	3 2.4
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	4 3.2
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	5 4.0
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	6 4.8
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	7 5.6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	8 6.4
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	9 7.2
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	7
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	1 0.7
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	2 1.4
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	3 2.1
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	4 2.8
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	5 3.5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	6 4.2
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	7 4.9
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	8 5.6
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	9 6.3
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	6
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	1 0.6
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	2 1.2
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	3 1.8
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	4 2.4
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	5 3.0
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	6 3.6
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	7 4.2
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	8 4.8
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	9 5.4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	
100	0000	0004	0009	0013	0017	0022	0026	0030	0035	0039	
N.	0	1	2	3	4	5	6	7	8	9	

Table I. Logarithms of Numbers

No.	0	1	2	3	4	5	6	7	8	9	Prop. Parts
100	0000	0004	0009	0013	0017	0022	0026	0030	0035	0039	
101	0043	0048	0052	0056	0060	0065	0069	0073	0077	0082	
102	0086	0090	0095	0099	0103	0107	0111	0116	0120	0124	
103	0128	0133	0137	0141	0145	0149	0154	0158	0162	0166	
104	0170	0175	0179	0183	0187	0191	0195	0199	0204	0208	
105	0212	0216	0220	0224	0228	0233	0237	0241	0245	0249	
106	0253	0257	0261	0265	0269	0273	0278	0282	0286	0290	5
107	0294	0298	0302	0306	0310	0314	0318	0322	0326	0330	1 0.5
108	0334	0338	0342	0346	0350	0354	0358	0362	0366	0370	2 1.0
109	0374	0378	0382	0386	0390	0394	0398	0402	0406	0410	3 1.5
110	0414	0418	0422	0426	0430	0434	0438	0441	0445	0449	4 2.0
111	0453	0457	0461	0465	0469	0473	0477	0481	0484	0488	5 2.5
112	0492	0496	0500	0504	0508	0512	0515	0519	0523	0527	6 3.0
113	0531	0535	0538	0542	0546	0550	0554	0558	0561	0565	7 3.5
114	0569	0573	0577	0580	0584	0588	0592	0596	0599	0603	8 4.0
115	0607	0611	0615	0618	0622	0626	0630	0633	0637	0641	9 4.5
116	0645	0648	0652	0656	0660	0663	0667	0671	0674	0678	
117	0682	0686	0689	0693	0697	0700	0704	0708	0711	0715	4
118	0719	0722	0726	0730	0734	0737	0741	0745	0748	0752	1 0.4
119	0755	0759	0763	0766	0770	0774	0777	0781	0785	0788	2 0.8
120	0792	0795	0799	0803	0806	0810	0813	0817	0821	0824	3 1.2
121	0828	0831	0835	0839	0842	0846	0849	0853	0856	0860	4 1.6
122	0864	0867	0871	0874	0878	0881	0885	0888	0892	0896	5 2.0
123	0899	0903	0906	0910	0913	0917	0920	0924	0927	0931	6 2.4
124	0934	0938	0941	0945	0948	0952	0955	0959	0962	0966	7 2.8
125	0969	0973	0976	0980	0983	0986	0990	0993	0997	1000	8 3.2
126	1004	1007	1011	1014	1017	1021	1024	1028	1031	1035	9 3.6
127	1038	1041	1045	1048	1052	1055	1059	1062	1065	1069	
128	1072	1075	1079	1082	1086	1089	1092	1096	1099	1103	3
129	1106	1109	1113	1116	1119	1123	1126	1129	1133	1136	1 0 3
130	1139	1143	1146	1149	1153	1156	1159	1163	1166	1169	2 0.6
131	1173	1176	1179	1183	1186	1189	1193	1196	1199	1202	3 0.9
132	1206	1209	1212	1216	1219	1222	1225	1229	1232	1235	4 1.2
133	1239	1242	1245	1248	1252	1255	1258	1261	1265	1268	5 1.5
134	1271	1274	1278	1281	1284	1287	1290	1294	1297	1300	6 1.8
135	1303	1307	1310	1313	1316	1319	1323	1326	1329	1332	7 2.1
136	1335	1339	1342	1345	1348	1351	1355	1358	1361	1364	8 2.4
137	1367	1370	1374	1377	1380	1383	1386	1389	1392	1396	9 2.7
138	1399	1402	1405	1408	1411	1414	1418	1421	1424	1427	
139	1430	1433	1436	1440	1443	1446	1449	1452	1455	1458	2
140	1461	1464	1467	1471	1474	1477	1480	1483	1486	1489	1 0.2
141	1492	1495	1498	1501	1504	1508	1511	1514	1517	1520	2 0.4
142	1523	1526	1529	1532	1535	1538	1541	1544	1547	1550	3 0.6
143	1553	1556	1559	1562	1565	1569	1572	1575	1578	1581	4 0.8
144	1584	1587	1590	1593	1596	1599	1602	1605	1608	1611	5 1.0
145	1614	1617	1620	1623	1626	1629	1632	1635	1638	1641	6 1.2
146	1644	1647	1649	1652	1655	1658	1661	1664	1667	1670	7 1.4
147	1673	1676	1679	1682	1685	1688	1691	1694	1697	1700	8 1.6
148	1703	1706	1708	1711	1714	1717	1720	1723	1726	1729	9 1.8
149	1732	1735	1738	1741	1744	1746	1749	1752	1755	1758	
150	1761	1764	1767	1770	1772	1775	1778	1781	1784	1787	
No.	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	

Table I. Logarithms of Numbers

N.	0	1	2	3	4	5	6	7	8	9	Prop. Parts
150	1761	1764	1767	1770	1772	1775	1778	1781	1784	1787	
151	1790	1793	1796	1798	1801	1804	1807	1810	1813	1816	
152	1818	1821	1824	1827	1830	1833	1836	1838	1841	1844	
153	1847	1850	1853	1855	1858	1861	1864	1867	1870	1872	
154	1875	1878	1881	1884	1886	1889	1892	1895	1898	1901	
155	1903	1906	1909	1912	1915	1917	1920	1923	1926	1928	
156	1931	1934	1937	1940	1942	1945	1948	1951	1953	1956	3
											1 0.3
157	1959	1962	1965	1967	1970	1973	1976	1978	1981	1984	2 0.6
158	1987	1989	1992	1995	1998	2000	2003	2006	2009	2011	3 0.9
159	2014	2017	2019	2022	2025	2028	2030	2033	2036	2038	4 1.2
160	2041	2044	2047	2049	2052	2055	2057	2060	2063	2066	5 1.5
161	2068	2071	2074	2076	2079	2082	2084	2087	2090	2092	6 1.8
162	2095	2098	2101	2103	2106	2109	2111	2114	2117	2119	7 2.1
163	2122	2125	2127	2130	2133	2135	2138	2140	2143	2146	8 2.4
											9 2.7
164	2148	2151	2154	2156	2159	2162	2164	2167	2170	2172	
165	2175	2177	2180	2183	2185	2188	2191	2193	2196	2198	
166	2201	2204	2206	2209	2212	2214	2217	2219	2222	2225	
167	2227	2230	2232	2235	2238	2240	2243	2245	2248	2251	
168	2253	2256	2258	2261	2263	2266	2269	2271	2274	2276	
169	2279	2281	2284	2287	2289	2292	2294	2297	2299	2302	
170	2304	2307	2310	2312	2315	2317	2320	2322	2325	2327	
171	2330	2333	2335	2338	2340	2343	2345	2348	2350	2353	
172	2355	2358	2360	2363	2365	2368	2370	2373	2375	2378	
173	2380	2383	2385	2388	2390	2393	2395	2398	2400	2403	
174	2405	2408	2410	2413	2415	2418	2420	2423	2425	2428	
175	2430	2433	2435	2438	2440	2443	2445	2448	2450	2453	
176	2455	2458	2460	2463	2465	2467	2470	2472	2475	2477	2
											1 0.2
177	2480	2482	2485	2487	2490	2492	2494	2497	2499	2502	2 0.4
178	2504	2507	2509	2512	2514	2516	2519	2521	2524	2526	3 0.6
179	2529	2531	2533	2536	2538	2541	2543	2545	2548	2550	4 0.8
180	2553	2555	2558	2560	2562	2565	2567	2570	2572	2574	5 1.0
181	2577	2579	2582	2584	2586	2589	2591	2594	2596	2598	6 1.2
182	2601	2603	2605	2608	2610	2613	2615	2617	2620	2622	7 1.4
183	2625	2627	2629	2632	2634	2636	2639	2641	2643	2646	8 1.6
											9 1.8
184	2648	2651	2653	2655	2658	2660	2662	2665	2667	2669	
185	2672	2674	2676	2679	2681	2683	2686	2688	2690	2693	
186	2695	2697	2700	2702	2704	2707	2709	2711	2714	2716	
187	2718	2721	2723	2725	2728	2730	2732	2735	2737	2739	
188	2742	2744	2746	2749	2751	2753	2755	2758	2760	2762	
189	2765	2767	2769	2772	2774	2776	2778	2781	2783	2785	
190	2788	2790	2792	2794	2797	2799	2801	2804	2806	2808	
191	2810	2813	2815	2817	2819	2822	2824	2826	2828	2831	
192	2833	2835	2838	2840	2842	2844	2847	2849	2851	2853	
193	2856	2858	2860	2862	2865	2867	2869	2871	2874	2876	
194	2878	2880	2882	2885	2887	2889	2891	2894	2896	2898	
195	2900	2903	2905	2907	2909	2911	2914	2916	2918	2920	
196	2923	2925	2927	2929	2931	2934	2936	2938	2940	2942	
197	2945	2947	2949	2951	2953	2956	2958	2960	2962	2964	
198	2967	2969	2971	2973	2975	2978	2980	2982	2984	2986	
199	2989	2991	2993	2995	2997	2999	3002	3004	3006	3008	
200	3010	3012	3015	3017	3019	3021	3023	3025	3028	3030	
N.	0	1	2	3	4	5	6	7	8	9	

Table II. Values and Logarithms of Trigonometric Functions

[Characteristics of Logarithms omitted — determine by the usual rule from the value]

Radians	Degrees	Sine Value	Sine $\log_{10}$	Tangent Value	Tangent $\log_{10}$	Cotangent Value	Cotangent $\log_{10}$	Cosine Value	Cosine $\log_{10}$		
.0000	0° 00′	.0000	—	.0000	—	—	—	1.0000	.0000	90° 00′	1.5708
.0029	10	.0029	.4637	.0029	.4637	343.77	.5363	1.0000	.0000	50	1.5679
.0058	20	.0058	.7648	.0058	.7648	171.89	.2352	1.0000	.0000	40	1.5650
.0087	30	.0087	.9408	.0087	.9409	114.59	.0591	1.0000	.0000	30	1.5621
.0116	40	.0116	.0658	.0116	.0658	85.940	.9342	.9999	.0000	20	1.5592
.0145	50	.0145	.1627	.0145	.1627	68.750	.8373	.9999	.0000	10	1.5563
.0175	1° 00′	.0175	.2419	.0175	.2419	57.290	.7581	.9998	.9999	89° 00′	1.5533
.0204	10	.0204	.3088	.0204	.3089	49.104	.6911	.9998	.9999	50	1.5504
.0233	20	.0233	.3668	.0233	.3669	42.964	.6331	.9997	.9999	40	1.5475
.0262	30	.0262	.4179	.0262	.4181	38.188	.5819	.9997	.9999	30	1.5446
.0291	40	.0291	.4637	.0291	.4638	34.368	.5362	.9996	.9998	20	1.5417
.0320	50	.0320	.5050	.0320	.5053	31.242	.4947	.9995	.9998	10	1.5388
.0349	2° 00′	.0349	.5428	.0349	.5431	28.636	.4569	.9994	.9997	88° 00′	1.5359
.0378	10	.0378	.5776	.0378	.5779	26.432	.4221	.9993	.9997	50	1.5330
.0407	20	.0407	.6097	.0407	.6101	24.542	.3899	.9992	.9996	40	1.5301
.0436	30	.0436	.6397	.0437	.6401	22.904	.3599	.9990	.9996	30	1.5272
.0465	40	.0465	.6677	.0466	.6682	21.470	.3318	.9989	.9995	20	1.5243
.0495	50	.0494	.6940	.0495	.6945	20.206	.3055	.9988	.9995	10	1.5213
.0524	3° 00′	.0523	.7188	.0524	.7194	19.081	.2806	.9986	.9994	87° 00′	1.5184
.0553	10	.0552	.7423	.0553	.7429	18.075	.2571	.9985	.9993	50	1.5155
.0582	20	.0581	.7645	.0582	.7652	17.169	.2348	.9983	.9993	40	1.5126
.0611	30	.0610	.7857	.0612	.7865	16.350	.2135	.9981	.9992	30	1.5097
.0640	40	.0640	.8059	.0641	.8067	15.605	.1933	.9980	.9991	20	1.5068
.0669	50	.0669	.8251	.0670	.8261	14.924	.1739	.9978	.9990	10	1.5039
.0698	4° 00′	.0698	.8436	.0699	.8446	14.301	.1554	.9976	.9989	86° 00′	1.5010
.0727	10	.0727	.8613	.0729	.8624	13.727	.1376	.9974	.9989	50	1.4981
.0756	20	.0756	.8783	.0758	.8795	13.197	.1205	.9971	.9988	40	1.4952
.0785	30	.0785	.8946	.0787	.8960	12.706	.1040	.9969	.9987	30	1.4923
.0814	40	.0814	.9104	.0816	.9118	12.251	.0882	.9967	.9986	20	1.4893
.0844	50	.0843	.9256	.0846	.9272	11.826	.0728	.9964	.9985	10	1.4864
.0873	5° 00′	.0872	.9403	.0875	.9420	11.430	.0580	.9962	.9983	85° 00′	1.4835
.0902	10	.0901	.9545	.0904	.9563	11.059	.0437	.9959	.9982	50	1.4806
.0931	20	.0929	.9682	.0934	.9701	10.712	.0299	.9957	.9981	40	1.4777
.0960	30	.0958	.9816	.0963	.9836	10.385	.0164	.9954	.9980	30	1.4748
.0989	40	.0987	.9945	.0992	.9966	10.078	.0034	.9951	.9979	20	1.4719
.1018	50	.1016	.0070	.1022	.0093	9.7882	.9907	.9948	.9977	10	1.4690
.1047	6° 00′	.1045	.0192	.1051	.0216	9.5144	.9784	.9945	.9976	84° 00′	1.4661
.1076	10	.1074	.0311	.1080	.0336	9.2553	.9664	.9942	.9975	50	1.4632
.1105	20	.1103	.0426	.1110	.0453	9.0098	.9547	.9939	.9973	40	1.4603
.1134	30	.1132	.0539	.1139	.0567	8.7769	.9433	.9936	.9972	30	1.4573
.1164	40	.1161	.0648	.1169	.0678	8.5555	.9322	.9932	.9971	20	1.4544
.1193	50	.1190	.0755	.1198	.0786	8.3450	.9214	.9929	.9969	10	1.4515
.1222	7° 00′	.1219	.0859	.1228	.0891	8.1443	.9109	.9925	.9968	83° 00′	1.4486
.1251	10	.1248	.0961	.1257	.0995	7.9530	.9005	.9922	.9966	50	1.4457
.1280	20	.1276	.1060	.1287	.1096	7.7704	.8904	.9918	.9964	40	1.4428
.1309	30	.1305	.1157	.1317	.1194	7.5958	.8806	.9914	.9963	30	1.4399
.1338	40	.1334	.1252	.1346	.1291	7.4287	.8709	.9911	.9961	20	1.4370
.1367	50	.1363	.1345	.1376	.1385	7.2687	.8615	.9907	.9959	10	1.4341
.1396	8° 00′	.1392	.1436	.1405	.1478	7.1154	.8522	.9903	.9958	82° 00′	1.4312
.1425	10	.1421	.1525	.1435	.1569	6.9682	.8431	.9899	.9956	50	1.4283
.1454	20	.1449	.1612	.1465	.1658	6.8269	.8342	.9894	.9954	40	1.4254
.1484	30	.1478	.1697	.1495	.1745	6.6912	.8255	.9890	.9952	30	1.4224
.1513	40	.1507	.1781	.1524	.1831	6.5606	.8169	.9886	.9950	20	1.4195
.1542	50	.1536	.1863	.1554	.1915	6.4348	.8085	.9881	.9948	10	1.4166
.1571	9° 00′	.1564	.1943	.1584	.1997	6.3138	.8003	.9877	.9946	81° 00′	1.4137
		Cosine Value	Cosine $\log_{10}$	Cotangent Value	Cotangent $\log_{10}$	Tangent Value	Tangent $\log_{10}$	Sine Value	Sine $\log_{10}$	Degrees	Radians

Table II. Values and Logarithms of Trigonometric Functions

[Characteristics of Logarithms omitted — determine by the usual rule from the value]

Radians	Degrees	Sine Value	Sine $\log_{10}$	Tangent Value	Tangent $\log_{10}$	Cotangent Value	Cotangent $\log_{10}$	Cosine Value	Cosine $\log_{10}$		
.1571	9° 00′	.1564	.1943	.1584	.1997	6.3138	.8003	.9877	.9946	81° 00′	1.4137
.1600	10	.1593	.2022	.1614	.2078	6.1970	.7922	.9872	.9944	50	1.4108
.1629	20	.1622	.2100	.1644	.2158	6.0844	.7842	.9868	.9942	40	1.4079
.1658	30	.1650	.2176	.1673	.2236	5.9758	.7764	.9863	.9940	30	1.4050
.1687	40	.1679	.2251	.1703	.2313	5.8708	.7687	.9858	.9938	20	1.4021
.1716	50	.1708	.2324	.1733	.2389	5.7694	.7611	.9853	.9936	10	1.3992
.1745	10° 00′	.1736	.2397	.1763	.2463	5.6713	.7537	.9848	.9934	80° 00′	1.3963
.1774	10	.1765	.2468	.1793	.2536	5.5764	.7464	.9843	.9931	50	1.3934
.1804	20	.1794	.2538	.1823	.2609	5.4845	.7391	.9838	.9929	40	1.3904
.1833	30	.1822	.2606	.1853	.2680	5.3955	.7320	.9833	.9927	30	1.3875
.1862	40	.1851	.2674	.1883	.2750	5.3093	.7250	.9827	.9924	20	1.3846
.1891	50	.1880	.2740	.1914	.2819	5.2257	.7181	.9822	.9922	10	1.3817
.1920	11° 00′	.1908	.2806	.1944	.2887	5.1446	.7113	.9816	.9919	79° 00′	1.3788
.1949	10	.1937	.2870	.1974	.2953	5.0658	.7047	.9811	.9917	50	1.3759
.1978	20	.1965	.2934	.2004	.3020	4.9894	.6980	.9805	.9914	40	1.3730
.2007	30	.1994	.2997	.2035	.3085	4.9152	.6915	.9799	.9912	30	1.3701
.2036	40	.2022	.3058	.2065	.3149	4.8430	.6851	.9793	.9909	20	1.3672
.2065	50	.2051	.3119	.2095	.3212	4.7729	.6788	.9787	.9907	10	1.3643
.2094	12° 00′	.2079	.3179	.2126	.3275	4.7046	.6725	.9781	.9904	78° 00′	1.3614
.2123	10	.2108	.3238	.2156	.3336	4.6382	.6664	.9775	.9901	50	1.3584
.2153	20	.2136	.3296	.2186	.3397	4.5736	.6603	.9769	.9899	40	1.3555
.2182	30	.2164	.3353	.2217	.3458	4.5107	.6542	.9763	.9896	30	1.3526
.2211	40	.2193	.3410	.2247	.3517	4.4494	.6483	.9757	.9893	20	1.3497
.2240	50	.2221	.3466	.2278	.3576	4.3897	.6424	.9750	.9890	10	1.3468
.2269	13° 00′	.2250	.3521	.2309	.3634	4.3315	.6366	.9744	.9887	77° 00′	1.3439
.2298	10	.2278	.3575	.2339	.3691	4.2747	.6309	.9737	.9884	50	1.3410
.2327	20	.2306	.3629	.2370	.3748	4.2193	.6252	.9730	.9881	40	1.3381
.2356	30	.2334	.3682	.2401	.3804	4.1653	.6196	.9724	.9878	30	1.3352
.2385	40	.2363	.3734	.2432	.3859	4.1126	.6141	.9717	.9875	20	1.3323
.2414	50	.2391	.3786	.2462	.3914	4.0611	.6086	.9710	.9872	10	1.3294
.2443	14° 00′	.2419	.3837	.2493	.3968	4.0108	.6032	.9703	.9869	76° 00′	1.3265
.2473	10	.2447	.3887	.2524	.4021	3.9617	.5979	.9696	.9866	50	1.3235
.2502	20	.2476	.3937	.2555	.4074	3.9136	.5926	.9689	.9863	40	1.3206
.2531	30	.2504	.3986	.2586	.4127	3.8667	.5873	.9681	.9859	30	1.3177
.2560	40	.2532	.4035	.2617	.4178	3.8208	.5822	.9674	.9856	20	1.3148
.2589	50	.2560	.4083	.2648	.4230	3.7760	.5770	.9667	.9853	10	1.3119
.2618	15° 00′	.2588	.4130	.2679	.4281	3.7321	.5719	.9659	.9849	75° 00′	1.3090
.2647	10	.2616	.4177	.2711	.4331	3.6891	.5669	.9652	.9846	50	1.3061
.2676	20	.2644	.4223	.2742	.4381	3.6470	.5619	.9644	.9843	40	1.3032
.2705	30	.2672	.4269	.2773	.4430	3.6059	.5570	.9636	.9839	30	1.3003
.2734	40	.2700	.4314	.2805	.4479	3.5656	.5521	.9628	.9836	20	1.2974
.2763	50	.2728	.4359	.2836	.4527	3.5261	.5473	.9621	.9832	10	1.2945
.2793	16° 00′	.2756	.4403	.2867	.4575	3.4874	.5425	.9613	.9828	74° 00′	1.2915
.2822	10	.2784	.4447	.2899	.4622	3.4495	.5378	.9605	.9825	50	1.2886
.2851	20	.2812	.4491	.2931	.4669	3.4124	.5331	.9596	.9821	40	1.2857
.2880	30	.2840	.4533	.2962	.4716	3.3759	.5284	.9588	.9817	30	1.2828
.2909	40	.2868	.4576	.2994	.4762	3.3402	.5238	.9580	.9814	20	1.2799
.2938	50	.2896	.4618	.3026	.4808	3.3052	.5192	.9572	.9810	10	1.2770
.2967	17° 00′	.2924	.4659	.3057	.4853	3.2709	.5147	.9563	.9806	73° 00′	1.2741
.2996	10	.2952	.4700	.3089	.4898	3.2371	.5102	.9555	.9802	50	1.2712
.3025	20	.2979	.4741	.3121	.4943	3.2041	.5057	.9546	.9798	40	1.2683
.3054	30	.3007	.4781	.3153	.4987	3.1716	.5013	.9537	.9794	30	1.2654
.3083	40	.3035	.4821	.3185	.5031	3.1397	.4969	.9528	.9790	20	1.2625
.3113	50	.3062	.4861	.3217	.5075	3.1084	.4925	.9520	.9786	10	1.2595
.3142	18° 00′	.3090	.4900	.3249	.5118	3.0777	.4882	.9511	.9782	72° 00′	1.2566
		Value	$\log_{10}$	Value	$\log_{10}$	Value	$\log_{10}$	Value	$\log_{10}$	Degrees	Radians
		Cosine		Cotangent		Tangent		Sine			

Table II. Values and Logarithms of Trigonometric Functions

[Characteristics of Logarithms omitted — determine by the usual rule from the value]

Radians	Degrees	Sine Value	Sine $\log_{10}$	Tangent Value	Tangent $\log_{10}$	Cotangent Value	Cotangent $\log_{10}$	Cosine Value	Cosine $\log_{10}$		
.3142	18° 00′	.3090	.4900	.3249	.5118	3.0777	.4882	.9511	.9782	72° 00′	1.2566
.3171	10	.3118	.4939	.3281	.5161	3.0475	.4839	.9502	.9778	50	1.2537
.3200	20	.3145	.4977	.3314	.5203	3.0178	.4797	.9492	.9774	40	1.2508
.3229	30	.3173	.5015	.3346	.5245	2.9887	.4755	.9483	.9770	30	1.2479
.3258	40	.3201	.5052	.3378	.5287	2.9600	.4713	.9474	.9765	20	1.2450
.3287	50	.3228	.5090	.3411	.5329	2.9319	.4671	.9465	.9761	10	1.2421
.3316	19° 00′	.3256	.5126	.3443	.5370	2.9042	.4630	.9455	.9757	71° 00′	1.2392
.3345	10	.3283	.5163	.3476	.5411	2.8770	.4589	.9446	.9752	50	1.2363
.3374	20	.3311	.5199	.3508	.5451	2.8502	.4549	.9436	.9748	40	1.2334
.3403	30	.3338	.5235	.3541	.5491	2.8239	.4509	.9426	.9743	30	1.2305
.3432	40	.3365	.5270	.3574	.5531	2.7980	.4469	.9417	.9739	20	1.2275
.3462	50	.3393	.5306	.3607	.5571	2.7725	.4429	.9407	.9734	10	1.2246
.3491	20° 00′	.3420	.5341	.3640	.5611	2.7475	.4389	.9397	.9730	70° 00′	1.2217
.3520	10	.3448	.5375	.3673	.5650	2.7228	.4350	.9387	.9725	50	1.2188
.3549	20	.3475	.5409	.3706	.5689	2.6985	.4311	.9377	.9721	40	1.2159
.3578	30	.3502	.5443	.3739	.5727	2.6746	.4273	.9367	.9716	30	1.2130
.3607	40	.3529	.5477	.3772	.5766	2.6511	.4234	.9356	.9711	20	1.2101
.3636	50	.3557	.5510	.3805	.5804	2.6279	.4196	.9346	.9706	10	1.2072
.3665	21° 00′	.3584	.5543	.3839	.5842	2.6051	.4158	.9336	.9702	69° 00′	1.2043
.3694	10	.3611	.5576	.3872	.5879	2.5826	.4121	.9325	.9697	50	1.2014
.3723	20	.3638	.5609	.3906	.5917	2.5605	.4083	.9315	.9692	40	1.1985
.3752	30	.3665	.5641	.3939	.5954	2.5386	.4046	.9304	.9687	30	1.1956
.3782	40	.3692	.5673	.3973	.5991	2.5172	.4009	.9293	.9682	20	1.1926
.3811	50	.3719	.5704	.4006	.6028	2.4960	.3972	.9283	.9677	10	1.1897
.3840	22° 00′	.3746	.5736	.4040	.6064	2.4751	.3936	.9272	.9672	68° 00′	1.1868
.3869	10	.3773	.5767	.4074	.6100	2.4545	.3900	.9261	.9667	50	1.1839
.3898	20	.3800	.5798	.4108	.6136	2.4342	.3864	.9250	.9661	40	1.1810
.3927	30	.3827	.5828	.4142	.6172	2.4142	.3828	.9239	.9656	30	1.1781
.3956	40	.3854	.5859	.4176	.6208	2.3945	.3792	.9228	.9651	20	1.1752
.3985	50	.3881	.5889	.4210	.6243	2.3750	.3757	.9216	.9646	10	1.1723
.4014	23° 00′	.3907	.5919	.4245	.6279	2.3559	.3721	.9205	.9640	67° 00′	1.1694
.4043	10	.3934	.5948	.4279	.6314	2.3369	.3686	.9194	.9635	50	1.1665
.4072	20	.3961	.5978	.4314	.6348	2.3183	.3652	.9182	.9629	40	1.1636
.4102	30	.3987	.6007	.4348	.6383	2.2998	.3617	.9171	.9624	30	1.1606
.4131	40	.4014	.6036	.4383	.6417	2.2817	.3583	.9159	.9618	20	1.1577
.4160	50	.4041	.6065	.4417	.6452	2.2637	.3548	.9147	.9613	10	1.1548
.4189	24° 00′	.4067	.6093	.4452	.6486	2.2460	.3514	.9135	.9607	66° 00′	1.1519
.4218	10	.4094	.6121	.4487	.6520	2.2286	.3480	.9124	.9602	50	1.1490
.4247	20	.4120	.6149	.4522	.6553	2.2113	.3447	.9112	.9596	40	1.1461
.4276	30	.4147	.6177	.4557	.6587	2.1943	.3413	.9100	.9590	30	1.1432
.4305	40	.4173	.6205	.4592	.6620	2.1775	.3380	.9088	.9584	20	1.1403
.4334	50	.4200	.6232	.4628	.6654	2.1609	.3346	.9075	.9579	10	1.1374
.4363	25° 00′	.4226	.6259	.4663	.6687	2.1445	.3313	.9063	.9573	65° 00′	1.1345
.4392	10	.4253	.6286	.4699	.6720	2.1283	.3280	.9051	.9567	50	1.1316
.4422	20	.4279	.6313	.4734	.6752	2.1123	.3248	.9038	.9561	40	1.1286
.4451	30	.4305	.6340	.4770	.6785	2.0965	.3215	.9026	.9555	30	1.1257
.4480	40	.4331	.6366	.4806	.6817	2.0809	.3183	.9013	.9549	20	1.1228
.4509	50	.4358	.6392	.4841	.6850	2.0655	.3150	.9001	.9543	10	1.1199
.4538	26° 00′	.4384	.6418	.4877	.6882	2.0503	.3118	.8988	.9537	64° 00′	1.1170
.4567	10	.4410	.6444	.4913	.6914	2.0353	.3086	.8975	.9530	50	1.1141
.4596	20	.4436	.6470	.4950	.6946	2.0204	.3054	.8962	.9524	40	1.1112
.4625	30	.4462	.6495	.4986	.6977	2.0057	.3023	.8949	.9518	30	1.1083
.4654	40	.4488	.6521	.5022	.7009	1.9912	.2991	.8936	.9512	20	1.1054
.4683	50	.4514	.6546	.5059	.7040	1.9768	.2960	.8923	.9505	10	1.1025
.4712	27° 00′	.4540	.6570	.5095	.7072	1.9626	.2928	.8910	.9499	63° 00′	1.0996
		Value	Cosine $\log_{10}$	Value	Cotangent $\log_{10}$	Value	Tangent $\log_{10}$	Value	Sine $\log_{10}$	Degrees	Radians

Table II. Values and Logarithms of Trigonometric Functions

[Characteristics of Logarithms omitted — determine by the usual rule from the value]

Radians	Degrees	Sine Value	Sine $\log_{10}$	Tangent Value	Tangent $\log_{10}$	Cotangent Value	Cotangent $\log_{10}$	Cosine Value	Cosine $\log_{10}$		
.4712	27° 00′	.4540	.6570	.5095	.7072	1.9626	.2928	.8910	.9499	63° 00′	1.0996
.4741	10	.4566	.6595	.5132	.7103	1.9486	.2897	.8897	.9492	50	1.0966
.4771	20	.4592	.6620	.5169	.7134	1.9347	.2866	.8884	.9486	40	1.0937
.4800	30	.4617	.6644	.5206	.7165	1.9210	.2835	.8870	.9479	30	1.0908
.4829	40	.4643	.6668	.5243	.7196	1.9074	.2804	.8857	.9473	20	1.0879
.4858	50	.4669	.6692	.5280	.7226	1.8940	.2774	.8843	.9466	10	1.0850
.4887	28° 00′	.4695	.6716	.5317	.7257	1.8807	.2743	.8829	.9459	62° 00′	1.0821
.4916	10	.4720	.6740	.5354	.7287	1.8676	.2713	.8816	.9453	50	1.0792
.4945	20	.4746	.6763	.5392	.7317	1.8546	.2683	.8802	.9446	40	1.0763
.4974	30	.4772	.6787	.5430	.7348	1.8418	.2652	.8788	.9439	30	1.0734
.5003	40	.4797	.6810	.5467	.7378	1.8291	.2622	.8774	.9432	20	1.0705
.5032	50	.4823	.6833	.5505	.7408	1.8165	.2592	.8760	.9425	10	1.0676
.5061	29° 00′	.4848	.6856	.5543	.7438	1.8040	.2562	.8746	.9418	61° 00′	1.0647
.5091	10	.4874	.6878	.5581	.7467	1.7917	.2533	.8732	.9411	50	1.0617
.5120	20	.4899	.6901	.5619	.7497	1.7796	.2503	.8718	.9404	40	1.0588
.5149	30	.4924	.6923	.5658	.7526	1.7675	.2474	.8704	.9397	30	1.0559
.5178	40	.4950	.6946	.5696	.7556	1.7556	.2444	.8689	.9390	20	1.0530
.5207	50	.4975	.6968	.5735	.7585	1.7437	.2415	.8675	.9383	10	1.0501
.5236	30° 00′	.5000	.6990	.5774	.7614	1.7321	.2386	.8660	.9375	60° 00′	1.0472
.5265	10	.5025	.7012	.5812	.7644	1.7205	.2356	.8646	.9368	50	1.0443
.5294	20	.5050	.7033	.5851	.7673	1.7090	.2327	.8631	.9361	40	1.0414
.5323	30	.5075	.7055	.5890	.7701	1.6977	.2299	.8616	.9353	30	1.0385
.5352	40	.5100	.7076	.5930	.7730	1.6864	.2270	.8601	.9346	20	1.0356
.5381	50	.5125	.7097	.5969	.7759	1.6753	.2241	.8587	.9338	10	1.0327
.5411	31° 00′	.5150	.7118	.6009	.7788	1.6643	.2212	.8572	.9331	59° 00′	1.0297
.5440	10	.5175	.7139	.6048	.7816	1.6534	.2184	.8557	.9323	50	1.0268
.5469	20	.5200	.7160	.6088	.7845	1.6426	.2155	.8542	.9315	40	1.0239
.5498	30	.5225	.7181	.6128	.7873	1.6319	.2127	.8526	.9308	30	1.0210
.5527	40	.5250	.7201	.6168	.7902	1.6212	.2098	.8511	.9300	20	1.0181
.5556	50	.5275	.7222	.6208	.7930	1.6107	.2070	.8496	.9292	10	1.0152
.5585	32° 00′	.5299	.7242	.6249	.7958	1.6003	.2042	.8480	.9284	58° 00′	1.0123
.5614	10	.5324	.7262	.6289	.7986	1.5900	.2014	.8465	.9276	50	1.0094
.5643	20	.5348	.7282	.6330	.8014	1.5798	.1986	.8450	.9268	40	1.0065
.5672	30	.5373	.7302	.6371	.8042	1.5697	.1958	.8434	.9260	30	1.0036
.5701	40	.5398	.7322	.6412	.8070	1.5597	.1930	.8418	.9252	20	1.0007
.5730	50	.5422	.7342	.6453	.8097	1.5497	.1903	.8403	.9244	10	.9977
.5760	33° 00′	.5446	.7361	.6494	.8125	1.5399	.1875	.8387	.9236	57° 00′	.9948
.5789	10	.5471	.7380	.6536	.8153	1.5301	.1847	.8371	.9228	50	.9919
.5818	20	.5495	.7400	.6577	.8180	1.5204	.1820	.8355	.9219	40	.9890
.5847	30	.5519	.7419	.6619	.8208	1.5108	.1792	.8339	.9211	30	.9861
.5876	40	.5544	.7438	.6661	.8235	1.5013	.1765	.8323	.9203	20	.9832
.5905	50	.5568	.7457	.6703	.8263	1.4919	.1737	.8307	.9194	10	.9803
.5934	34° 00′	.5592	.7476	.6745	.8290	1.4826	.1710	.8290	.9186	56° 00′	.9774
.5963	10	.5616	.7494	.6787	.8317	1.4733	.1683	.8274	.9177	50	.9745
.5992	20	.5640	.7513	.6830	.8344	1.4641	.1656	.8258	.9169	40	.9716
.6021	30	.5664	.7531	.6873	.8371	1.4550	.1629	.8241	.9160	30	.9687
.6050	40	.5688	.7550	.6916	.8398	1.4460	.1602	.8225	.9151	20	.9657
.6080	50	.5712	.7568	.6959	.8425	1.4370	.1575	.8208	.9142	10	.9628
.6109	35° 00′	.5736	.7586	.7002	.8452	1.4281	.1548	.8192	.9134	55° 00′	.9599
.6138	10	.5760	.7604	.7046	.8479	1.4193	.1521	.8175	.9125	50	.9570
.6167	20	.5783	.7622	.7089	.8506	1.4106	.1494	.8158	.9116	40	.9541
.6196	30	.5807	.7640	.7133	.8533	1.4019	.1467	.8141	.9107	30	.9512
.6225	40	.5831	.7657	.7177	.8559	1.3934	.1441	.8124	.9098	20	.9483
.6254	50	.5854	.7675	.7221	.8586	1.3848	.1414	.8107	.9089	10	.9454
.6283	36° 00′	.5878	.7692	.7265	.8613	1.3764	.1387	.8090	.9080	54° 00′	.9425
		Value	Cosine $\log_{10}$	Value	Cotangent $\log_{10}$	Value	Tangent $\log_{10}$	Value	Sine $\log_{10}$	Degrees	Radians

Table II. Values and Logarithms of Trigonometric Functions

[Characteristics of Logarithms omitted — determine by the usual rule from the value]

Radians	Degrees	Sine Value	Sine $\log_{10}$	Tangent Value	Tangent $\log_{10}$	Cotangent Value	Cotangent $\log_{10}$	Cosine Value	Cosine $\log_{10}$		
.6283	**36° 00′**	.5878	.7692	.7265	.8613	1.3764	.1387	.8090	.9080	**54° 00′**	.9425
.6312	10	.5901	.7710	.7310	.8639	1.3680	.1361	.8073	.9070	50	.9396
.6341	20	.5925	.7727	.7355	.8666	1.3597	.1334	.8056	.9061	40	.9367
.6370	30	.5948	.7744	.7400	.8692	1.3514	.1308	.8039	.9052	30	.9338
.6400	40	.5972	.7761	.7445	.8718	1.3432	.1282	.8021	.9042	20	.9308
.6429	50	.5995	.7778	.7490	.8745	1.3351	.1255	.8004	.9033	10	.9279
.6458	**37° 00′**	.6018	.7795	.7536	.8771	1.3270	.1229	.7986	.9023	**53° 00′**	.9250
.6487	10	.6041	.7811	.7581	.8797	1.3190	.1203	.7969	.9014	50	.9221
.6516	20	.6065	.7828	.7627	.8824	1.3111	.1176	.7951	.9004	40	.9192
.6545	30	.6088	.7844	.7673	.8850	1.3032	.1150	.7934	.8995	30	.9163
.6574	40	.6111	.7861	.7720	.8876	1.2954	.1124	.7916	.8985	20	.9134
.6603	50	.6134	.7877	.7766	.8902	1.2876	.1098	.7898	.8975	10	.9105
.6632	**38° 00′**	.6157	.7893	.7813	.8928	1.2799	.1072	.7880	.8965	**52° 00′**	.9076
.6661	10	.6180	.7910	.7860	.8954	1.2723	.1046	.7862	.8955	50	.9047
.6690	20	.6202	.7926	.7907	.8980	1.2647	.1020	.7844	.8945	40	.9018
.6720	30	.6225	.7941	.7954	.9006	1.2572	.0994	.7826	.8935	30	.8988
.6749	40	.6248	.7957	.8002	.9032	1.2497	.0968	.7808	.8925	20	.8959
.6778	50	.6271	.7973	.8050	.9058	1.2423	.0942	.7790	.8915	10	.8930
.6807	**39° 00′**	.6293	.7989	.8098	.9084	1.2349	.0916	.7771	.8905	**51° 00′**	.8901
.6836	10	.6316	.8004	.8146	.9110	1.2276	.0890	.7753	.8895	50	.8872
.6865	20	.6338	.8020	.8195	.9135	1.2203	.0865	.7735	.8884	40	.8843
.6894	30	.6361	.8035	.8243	.9161	1.2131	.0839	.7716	.8874	30	.8814
.6923	40	.6383	.8050	.8292	.9187	1.2059	.0813	.7698	.8864	20	.8785
.6952	50	.6406	.8066	.8342	.9212	1.1988	.0788	.7679	.8853	10	.8756
.6981	**40° 00′**	.6428	.8081	.8391	.9238	1.1918	.0762	.7660	.8843	**50° 00′**	.8727
.7010	10	.6450	.8096	.8441	.9264	1.1847	.0736	.7642	.8832	50	.8698
.7039	20	.6472	.8111	.8491	.9289	1.1778	.0711	.7623	.8821	40	.8668
.7069	30	.6494	.8125	.8541	.9315	1.1708	.0685	.7604	.8810	30	.8639
.7098	40	.6517	.8140	.8591	.9341	1.1640	.0659	.7585	.8800	20	.8610
.7127	50	.6539	.8155	.8642	.9366	1.1571	.0634	.7566	.8789	10	.8581
.7156	**41° 00′**	.6561	.8169	.8693	.9392	1.1504	.0608	.7547	.8778	**49° 00′**	.8552
.7185	10	.6583	.8184	.8744	.9417	1.1436	.0583	.7528	.8767	50	.8523
.7214	20	.6604	.8198	.8796	.9443	1.1369	.0557	.7509	.8756	40	.8494
.7243	30	.6626	.8213	.8847	.9468	1.1303	.0532	.7490	.8745	30	.8465
.7272	40	.6648	.8227	.8899	.9494	1.1237	.0506	.7470	.8733	20	.8436
.7301	50	.6670	.8241	.8952	.9519	1.1171	.0481	.7451	.8722	10	.8407
.7330	**42° 00′**	.6691	.8255	.9004	.9544	1.1106	.0456	.7431	.8711	**48° 00′**	.8378
.7359	10	.6713	.8269	.9057	.9570	1.1041	.0430	.7412	.8699	50	.8348
.7389	20	.6734	.8283	.9110	.9595	1.0977	.0405	.7392	.8688	40	.8319
.7418	30	.6756	.8297	.9163	.9621	1.0913	.0379	.7373	.8676	30	.8290
.7447	40	.6777	.8311	.9217	.9646	1.0850	.0354	.7353	.8665	20	.8261
.7476	50	.6799	.8324	.9271	.9671	1.0786	.0329	.7333	.8653	10	.8232
.7505	**43° 00′**	.6820	.8338	.9325	.9697	1.0724	.0303	.7314	.8641	**47° 00′**	.8203
.7534	10	.6841	.8351	.9380	.9722	1.0661	.0278	.7294	.8629	50	.8174
.7563	20	.6862	.8365	.9435	.9747	1.0599	.0253	.7274	.8618	40	.8145
.7592	30	.6884	.8378	.9490	.9772	1.0538	.0228	.7254	.8606	30	.8116
.7621	40	.6905	.8391	.9545	.9798	1.0477	.0202	.7234	.8594	20	.8087
.7650	50	.6926	.8405	.9601	.9823	1.0416	.0177	.7214	.8582	10	.8058
.7679	**44° 00′**	.6947	.8418	.9657	.9848	1.0355	.0152	.7193	.8569	**46° 00′**	.8029
.7709	10	.6967	.8431	.9713	.9874	1.0295	.0126	.7173	.8557	50	.7999
.7738	20	.6988	.8444	.9770	.9899	1.0235	.0101	.7153	.8545	40	.7970
.7767	30	.7009	.8457	.9827	.9924	1.0176	.0076	.7133	.8532	30	.7941
.7796	40	.7030	.8469	.9884	.9949	1.0117	.0051	.7112	.8520	20	.7912
.7825	50	.7050	.8482	.9942	.9975	1.0058	.0025	.7092	.8507	10	.7883
.7854	**45° 00′**	.7071	.8495	1.0000	.0000	1.0000	.0000	.7071	.8495	**45° 00′**	.7854
		Value	Cosine $\log_{10}$	Value	Cotangent $\log_{10}$	Value	Tangent $\log_{10}$	Value	Sine $\log_{10}$	Degrees	Radians

Table III. Radian Measure—Trigonometric Functions

Rad.	Deg.	Min.	sin.	cos.	tan.	Rad.	Deg.	Min.	sin.	cos.	tan
0.0	0	0	0	1	0	3.2	183	20.8	−.058	−.998	.058
0.1	5	43.8	.100	.995	.100	3.3	189	4.6	−.158	−.987	.161
0.2	11	27.5	.199	.980	.203	3.4	194	48.3	−.255	−.967	.264
0.3	17	11.3	.296	.955	.309	3.5	200	32.1	−.351	−.936	.375
0.4	22	55.1	.389	.921	.423	3.6	206	15.9	−.443	−.897	.493
0.5	28	38.9	.479	.878	.546	3.7	211	59.7	−.530	−.848	.625
0.6	34	22.6	.565	.825	.684	3.8	217	43.4	−.612	−.791	.774
0.7	40	6.4	.644	.765	.842	3.9	223	27.2	−.688	−.726	.947
0.8	45	50.2	.717	.697	1.030	4.0	229	11.0	−.757	−.654	1.158
0.9	51	34.0	.783	.622	1.260	4.1	234	54.8	−.818	−.575	1.424
1.0	57	17.7	.841	.540	1.557	4.2	240	38.5	−.872	−.490	1.778
1.1	63	1.5	.891	.454	1.965	4.3	246	22.3	−.916	−.401	2.286
1.2	68	45.3	.932	.362	2.572	4.4	252	6.1	−.952	−.307	3.096
1.3	74	29.1	.964	.267	3.602	4.5	257	49.9	−.978	−.211	4.638
1.4	80	12.8	.985	.170	5.798	4.6	263	33.6	−.994	−.112	8.859
1.5	85	56.6	.997	.071	14.101	4.7	269	17.4	−1.00	−.012	80.713
1.6	91	40.4	1.000	−.029	−34.233	4.8	275	1.2	−.996	.088	−11.385
1.7	97	24.2	.992	−.129	− 7.700	4.9	280	45.0	−.982	.187	− 5.267
1.8	103	7.9	.974	−.227	− 4.286	5.0	286	28.6	−.959	.284	− 3.381
1.9	108	51.7	.946	−.323	− 2.927	5.1	292	12.5	−.926	.378	− 2.449
2.0	114	35.5	.909	−.416	− 2.185	5.2	297	56.3	−.883	.469	− 1.885
2.1	120	19.3	.863	−.505	− 1.710	5.3	303	40.1	−.832	.554	− 1.501
2.2	126	3.0	.808	−.588	− 1.374	5.4	309	23.8	−.773	.635	− 1.217
2.3	131	46.8	.746	−.666	− 1.119	5.5	315	7.6	−.706	.709	− .996
2.4	137	30.6	.675	−.737	− .917	5.6	320	51.4	−.631	.776	− .814
2.5	143	14.4	.598	−.801	− .747	5.7	326	35.2	−.551	.835	− .660
2.6	148	58.1	.516	−.857	− .602	5.8	332	18.9	−.465	.886	− .525
2.7	154	41.9	.427	−.904	− .473	5.9	338	2.7	−.374	.927	− .403
2.8	160	25.7	.335	−.942	− .356	6.0	343	46.5	−.279	.960	− .291
2.9	166	9.5	.239	−.971	− .246	6.1	349	30.3	−.182	.983	− .185
3.0	171	53.2	.141	−.990	− .143	6.2	355	14.0	−.083	.997	− .083
3.1	177	37.0	.042	−.999	− .042	6.3	360	57.8	+.017	1.000	+ .017

Table IV. Squares and Cubes Square Roots and Cube Roots

No.	Square	Cube	Square Root	Cube Root	No.	Square	Cube	Square Root	Cube Root
1	1	1	1.000	1.000	51	2,601	132,651	7.141	3.708
2	4	8	1.414	1.260	52	2,704	140,608	7.211	3.733
3	9	27	1.732	1.442	53	2,809	148,877	7.280	3.756
4	16	64	2.000	1.587	54	2,916	157,464	7.348	3.780
5	25	125	2.236	1.710	55	3,025	166,375	7.416	3.803
6	36	216	2.449	1.817	56	3,136	175,616	7.483	3.826
7	49	343	2.646	1.913	57	3,249	185,193	7.550	3.849
8	64	512	2.828	2.000	58	3,364	195,112	7.616	3.871
9	81	729	3.000	2.080	59	3,481	205,379	7.681	3.893
10	100	1,000	3.162	2.154	60	3,600	216,000	7.746	3.915
11	121	1,331	3.317	2.224	61	3,721	226,981	7.810	3.936
12	144	1,728	3.464	2.289	62	3,844	238,328	7.874	3.958
13	169	2,197	3.606	2.351	63	3,969	250,047	7.937	3.979
14	196	2,744	3.742	2.410	64	4,096	262,144	8.000	4.000
15	225	3,375	3.873	2.466	65	4,225	274,625	8.062	4.021
16	256	4,096	4.000	2.520	66	4,356	287,496	8.124	4.041
17	289	4,913	4.123	2.571	67	4,489	300,763	8.185	4.062
18	324	5,832	4.243	2.621	68	4,624	314,432	8.246	4.082
19	361	6,859	4.359	2.668	69	4,761	328,509	8.307	4.102
20	400	8,000	4.472	2.714	70	4,900	343,000	8.367	4.121
21	441	9,261	4.583	2.759	71	5,041	357,911	8.426	4.141
22	484	10,648	4.690	2.802	72	5,184	373,248	8.485	4.160
23	529	12,167	4.796	2.844	73	5,329	389,017	8.544	4.179
24	576	13,824	4.899	2.884	74	5,476	405,224	8.602	4.198
25	625	15,625	5.000	2.924	75	5,625	421,875	8.660	4.217
26	676	17,576	5.099	2.962	76	5,776	438,976	8.718	4.236
27	729	19,683	5.196	3.000	77	5,929	456,533	8.775	4.254
28	784	21,952	5.292	3.037	78	6,084	474,552	8.832	4.273
29	841	24,389	5.385	3.072	79	6,241	493,039	8.888	4.291
30	900	27,000	5.477	3.107	80	6,400	512,000	8.944	4.309
31	961	29,791	5.568	3.141	81	6,561	531,441	9.000	4.327
32	1,024	32,768	5.657	3.175	82	6,724	551,368	9.055	4.344
33	1,089	35,937	5.745	3.208	83	6,889	571,787	9.110	4.362
34	1,156	39,304	5.831	3.240	84	7,056	592,704	9.165	4.380
35	1,225	42,875	5.916	3.271	85	7,225	614,125	9.220	4.397
36	1,296	46,656	6.000	3.302	86	7,396	636,056	9.274	4.414
37	1,369	50,653	6.083	3.332	87	7,569	658,503	9.327	4.431
38	1,444	54,872	6.164	3.362	88	7,744	681,472	9.381	4.448
39	1,521	59,319	6.245	3.391	89	7,921	704,969	9.434	4.465
40	1,600	64,000	6.325	3.420	90	8,100	729,000	9.487	4.481
41	1,681	68,921	6.403	3.448	91	8,281	753,571	9.539	4.498
42	1,764	74,088	6.481	3.476	92	8,464	778,688	9.592	4.514
43	1,849	79,507	6.557	3.503	93	8,649	804,357	9.644	4.531
44	1,936	85,184	6.633	3.530	94	8,836	830,584	9.695	4.547
45	2,025	91,125	6.708	3.557	95	9,025	857,375	9.747	4.563
46	2,116	97,336	6.782	3.583	96	9,216	884,736	9.798	4.579
47	2,209	103,823	6.856	3.609	97	9,409	912,673	9.849	4.595
48	2,304	110,592	6.928	3.634	98	9,604	941,192	9.899	4.610
49	2,401	117,649	7.000	3.659	99	9,801	970,299	9.950	4.626
50	2,500	125,000	7.071	3.684	100	10,000	1,000,000	10.000	4.642

For a more complete table, see The Macmillan Tables, pp. 94–111.

Table V. Logarithms of Important Constants

N = Number	Value of N	$\log_{10} N$
π	3.14159265	0.49714987
$1 \div \pi$	0.31830989	9.50285013
π^2	9.86960440	0.99429975
$\sqrt{\pi}$	1.77245385	0.24857494
e = Napierian Base	2.71828183	0.43429448
$M = \log_{10} e$	0.43429448	9.63778431
$1 \div M = \log_e 10$	2.30258509	0.36221569
$180 \div \pi$ = degrees in 1 radian	57.2957795	1.75812262
$\pi \div 180$ = radians in 1°	0.01745329	8.24187738
$\pi \div 10800$ = radians in 1′	0.0002908882	6.4637261
$\pi \div 648000$ = radians in 1″	0.000004848136811095	4.68557487
sin 1″	0.000004848136811076	4.68557487
tan 1″	0.000004848136811152	4.68557487
centimeters in 1 ft.	30.480	1.4840158
feet in 1 cm.	0.032808	8.5159842
inches in 1 m.	39.37	1.5951654
pounds in 1 kg.	2.20462	0.3433340
kilograms in 1 lb.	0.453593	9.6566660
g	32.16 ft./sec./sec.	1.5073160
	= 981 cm./sec./sec.	2.9916690
weight of 1 cu. ft. of water	62.425 lb. (max. density)	1.7953586
weight of 1 cu. ft. of air	0.0807 lb. (at 32° F.)	8.9068735
cu. in. in 1 (U. S.) gallon	231.	2.3636120
ft. lb. per sec. in 1 H. P.	550.	2.7403627
kg. m. per sec. in 1 H. P.	76.0404	1.8810445
watts in 1 H. P.	745.957	2.8727135

Table VI. Degrees to Radians

1°	.01745	10°	.17453	100°	1.74533	6′	.00175	6″	.00003
2°	.03491	20°	.34907	110°	1.91986	7′	.00204	7″	.00003
3°	.05236	30°	.52360	120°	2.09440	8′	.00233	8″	.00004
4°	.06981	40°	.69813	130°	2.26893	9′	.00262	9″	.00004
5°	.08727	50°	.87266	140°	2.44346	10′	.00291	10″	.00005
6°	.10472	60°	1.04720	150°	2.61799	20′	.00582	20″	.00010
7°	.12217	70°	1.22173	160°	2.79253	30′	.00873	30″	.00015
8°	.13963	80°	1.39626	170°	2.96706	40′	.01164	40″	.00019
9°	.15708	90°	1.57080	180°	3.14159	50′	.01454	50″	.00024

Table VII. Compound Interest Table

Amount of One Dollar Principal with Compound Interest at Various Rates.

Years.	2½ Per Cent.	3 Per Cent.	3½ Per Cent.	4 Per Cent.	4½ Per Cent.	5 Per Cent.	5½ Per Cent.	6 Per Cent.	6½ Per Cent.	7 Per Cent.	8 Per Cent.
1	$1.025	$1.030	$1.035	$1.040	$1.045	$1.050	$1.055	$1.060	$1.065	$1.070	$1.800
2	1.051	1.061	1.071	1.082	1.092	1.103	1.113	1.124	1.134	1.145	1.166
3	1.077	1.093	1.109	1.125	1.141	1.158	1.174	1.191	1.208	1.225	1.260
4	1.104	1.126	1.148	1.170	1.193	1.216	1.239	1.262	1.286	1.311	1.360
5	1.131	1.159	1.188	1.217	1.246	1.276	1.307	1.338	1.370	1.403	1.469
6	1.160	1.194	1.229	1.265	1.302	1.340	1.379	1.419	1.459	1.501	1.587
7	1.189	1.230	1.272	1.316	1.361	1.407	1.455	1.504	1.554	1.606	1.714
8	1.218	1.267	1.317	1.369	1.422	1.477	1.535	1.594	1.655	1.718	1.851
9	1.249	1.305	1.363	1.423	1.486	1.551	1.619	1.689	1.763	1.838	1.999
10	1.280	1.344	1.411	1.480	1.553	1.629	1.708	1.791	1.877	1.967	2.159
11	1.312	1.384	1.460	1.539	1.623	1.710	1.802	1.898	1.999	2.105	2.332
12	1.345	1.426	1.511	1.601	1.696	1.796	1.901	2.012	2.129	2.252	2.518
13	1.379	1.469	1.564	1.665	1.772	1.886	2.006	2.133	2.267	2.410	2.720
14	1.413	1.513	1.619	1.732	1.852	1.980	2.116	2.261	2.415	2.579	2.937
15	1.448	1.558	1.675	1.801	1.935	2.079	2.232	2.397	2.572	2.759	3.172
16	1.485	1.605	1.734	1.873	2.022	2.183	2.355	2.540	2.739	2.952	3.426
17	1.522	1.653	1.795	1.948	2.113	2.292	2.485	2.693	2.917	3.159	3.700
18	1.560	1.702	1.857	2.026	2.208	2.407	2.621	2.854	3.107	3.380	3.996
19	1.599	1.754	1.923	2.107	2.308	2.527	2.766	3.026	3.309	3.617	4.316
20	1.639	1.806	1.990	2.191	2.412	2.653	2.918	3.207	3.524	3.870	4.661
21	1.680	1.860	2.059	2.279	2.520	2.786	3.078	3.400	3.753	4.141	5.034
22	1.722	1.916	2.132	2.370	2.634	2.925	3.248	3.604	3.997	4.430	5.437
23	1.765	1.974	2.206	2.465	2.752	3.072	3.426	3.820	4.256	4.741	5.871
24	1.809	2.033	2.283	2.563	2.876	3.225	3.615	4.049	4.533	5.072	6.341
25	1.854	2.094	2.363	2.666	3.005	3.386	3.813	4.292	4.828	5.427	6.848
26	1.900	2.157	2.446	2.772	3.141	3.556	4.023	4.549	5.142	5.807	7.396
27	1.948	2.221	2.532	2.883	3.282	3.733	4.244	4.822	5.476	6.214	7.988
28	1.996	2.288	2.620	2.999	3.430	3.920	4.478	5.112	5.832	6.649	8.627
29	2.046	2.357	2.712	3.119	3.584	4.116	4.724	5.418	6.211	7.114	9.317
30	2.098	2.427	2.807	3.243	3.745	4.322	4.984	5.743	6.614	7.612	10.063
31	2.150	2.500	2.905	3.373	3.914	4.538	5.258	6.088	7.044	8.145	10.868
32	2.204	2.575	3.007	3.508	4.090	4.765	5.547	6.453	7.502	8.715	11.737
33	2.259	2.652	3.112	3.648	4.274	5.003	5.852	6.841	7.990	9.325	12.676
34	2.315	2.732	3.221	3.794	4.466	5.253	6.174	7.251	8.509	9.978	13.690
35	2.373	2.814	3.334	3.946	4.667	5.516	6.514	7.686	9.062	10.677	14.785
36	2.433	2.898	3.450	4.104	4.877	5.792	6.872	8.147	9.651	11.424	15.968
37	2.493	2.985	3.571	4.268	5.097	6.081	7.250	8.636	10.279	12.224	17.246
38	2.556	3.075	3.696	4.439	5.326	6.385	7.649	9.154	10.947	13.079	18.625
39	2.620	3.167	3.825	4.616	5.566	6.705	8.069	9.704	11.658	13.995	20.115
40	2.685	3.262	3.959	4.801	5.816	7.040	8.513	10.286	12.416	14.974	21.725
41	2.752	3.360	4.098	4.993	6.078	7.392	8.982	10.903	13.223	16.023	23.462
42	2.821	3.461	4.241	5.193	6.352	7.762	9.476	11.557	14.083	17.144	25.339
43	2.892	3.565	4.390	5.400	6.637	8.150	9.997	12.250	14.998	18.344	27.367
44	2.964	3.671	4.543	5.617	6.936	8.557	10.547	12.985	15.973	19.628	29.556
45	3.038	3.782	4.702	5.841	7.248	8.985	11.127	13.765	17.011	21.002	31.920
46	3.114	3.895	4.867	6.075	7.574	9.434	11.739	14.590	18.117	22.473	34.474
47	3.192	4.012	5.037	6.318	7.915	9.906	12.384	15.466	19.294	24.046	37.232
48	3.271	4.132	5.214	6.571	8.271	10.401	13.065	16.394	20.549	25.729	40.211
49	3.353	4.256	5.396	6.833	8.644	10.921	13.784	17.378	21.884	27.530	43.427
50	3.437	4.384	5.585	7.107	9.033	11.467	14.542	18.420	23.307	29.457	46.902

Table VIII. American Experience Mortality Table

Based on 100,000 living at age 10.

At Age.	Number Surviving.	Deaths.	At Age.	Number Surviving.	Deaths.	At Age.	Number Surviving.	Deaths.	At Age.	Number Surviving.	Deaths.
10	100,000	749	35	81,822	732	60	57,917	1,546	85	5,485	1,292
11	99,251	746	36	81,090	737	61	56,371	1,628	86	4,193	1,114
12	98,505	743	37	80,353	742	62	54,743	1,713	87	3,079	933
13	97,762	740	38	79,611	749	63	53,030	1,800	88	2,146	744
14	97,022	737	39	78,862	756	64	51,230	1,889	89	1,402	555
15	96,285	735	40	78,106	765	65	49,341	1,980	90	847	385
16	95,550	732	41	77,341	774	66	47,361	2,070	91	462	246
17	94,818	729	42	76,567	785	67	45,291	2,158	92	216	137
18	94,089	727	43	75,782	797	68	43,133	2,243	93	79	58
19	93,362	725	44	74,985	812	69	40,890	2,321	94	21	18
20	92,637	723	45	74,173	828	70	38,569	2,391	95	3	3
21	91,914	722	46	73,345	848	71	36,178	2,448			
22	91,192	721	47	72,497	870	72	33,730	2,487			
23	90,471	720	48	71,627	896	73	31,243	2,505			
24	89,751	719	49	70,731	927	74	28,738	2,501			
25	89,032	718	50	69,804	962	75	26,237	2,476			
26	88,314	718	51	68,842	1,001	76	23,761	2,431			
27	87,596	718	52	67,841	1,044	77	21,330	2,369			
28	86,878	718	53	66,797	1,091	78	18,961	2,291			
29	86,160	719	54	65,706	1,143	79	16,670	2,196			
30	85,441	720	55	64,563	1,199	80	14,474	2,091			
31	84,721	721	56	63,364	1,260	81	12,383	1,964			
32	84,000	723	57	62,104	1,325	82	10,419	1,816			
33	83,277	726	58	60,779	1,394	83	8,603	1,648			
34	82,551	729	59	59,385	1,468	84	6,955	1,470			

Table IX. Heights and Weights of Men

Light-face figures are 20 per cent. under and over the average.

Ages.		20–24	25–29	30–34	35–39	40–44	45–49	50–54	55–60
Ft.	In.								
5	0	96	100	102	105	106	107	107	107
		120	**125**	**128**	**131**	**133**	**134**	**134**	**134**
		144	150	154	157	160	161	161	161
	1	98	101	103	105	107	109	109	109
		122	**126**	**129**	**131**	**134**	**136**	**136**	**136**
		146	151	155	157	161	163	163	163
	2	99	102	105	106	109	110	110	110
		124	**128**	**131**	**133**	**136**	**138**	**138**	**138**
		149	154	157	160	163	166	166	166
	3	102	105	107	109	111	113	113	113
		127	**131**	**134**	**136**	**139**	**141**	**141**	**141**
		152	157	161	163	167	169	169	169
	4	105	108	110	112	114	115	116	116
		131	**135**	**138**	**140**	**143**	**144**	**145**	**145**
		157	162	166	168	172	173	174	174
	5	107	110	113	114	117	118	119	119
		134	**138**	**141**	**143**	**146**	**147**	**149**	**149**
		161	166	169	172	175	176	179	179
	6	110	114	116	118	120	121	122	122
		138	**142**	**145**	**147**	**150**	**151**	**153**	**153**
		166	170	174	176	180	181	184	184
	7	114	118	120	122	124	125	126	126
		142	**147**	**150**	**152**	**155**	**156**	**158**	**158**
		170	176	180	182	186	187	190	190
Ft.	In.								
5	8	117	121	123	126	128	129	130	130
		146	**151**	**154**	**157**	**160**	**161**	**163**	**163**
		175	181	185	188	192	193	196	196
	9	120	124	127	130	132	133	134	134
		150	**155**	**159**	**162**	**165**	**166**	**167**	**168**
		180	186	191	194	198	199	200	202
	10	123	127	131	134	136	137	138	138
		154	**159**	**164**	**167**	**170**	**171**	**172**	**173**
		185	191	197	200	204	205	206	208
	11	127	131	135	138	140	142	142	142
		159	**164**	**169**	**173**	**175**	**177**	**177**	**178**
		191	197	203	208	210	212	212	214
6	0	132	136	140	143	144	146	146	146
		165	**170**	**175**	**179**	**180**	**183**	**182**	**183**
		198	204	210	215	216	220	218	220
	1	136	142	145	148	149	151	150	151
		170	**177**	**181**	**185**	**186**	**189**	**188**	**189**
		204	212	217	222	223	227	226	227
	2	141	147	150	154	155	157	155	155
		176	**184**	**188**	**192**	**194**	**196**	**194**	**194**
		211	221	226	230	233	235	233	233
	3	145	152	156	160	162	163	161	158
		181	**190**	**195**	**200**	**203**	**204**	**201**	**198**
		217	228	234	240	244	245	241	238

EXPLANATION OF TABLE II*

Values and Logarithms of Trigonometric Functions

1. Direct reading of the values. This table gives the sines, cosines, tangents and cotangents of the angles from 0° to 45°; and by a simple device, indicated by the printing, the values of these functions for angles from 45° to 90° may be read directly from the same table. For angles less than 45° read down the page, the degrees and minutes being found on the left; for angles greater than 45° read up the page the degrees and minutes being found on the right.

To find a function of an angle (such as 15° 27′, for example) we employ the process of interpolation. To illustrate, let us find tan 15° 27′. In the table we find tan 15° 20′ = .2742 and tan 15° 30′ = .2773; we know that tan 15° 27′ lies between these two numbers. The process of interpolation depends on the assumption that between 15° 20′ and 15° 30′ the tangent of the angle varies directly as the angle; while this assumption is not strictly true, it gives an approximation sufficiently accurate for a four-place table. Thus we should assume that tan 15° 25′ is halfway between .2742 and .2773. We may state the problem as follows: An increase of 10′ in the angle increases the tangent .0031; assuming that the tangent varies as the angle, an increase of 7′ in the angle will increase the tangent by .7 × .0031 = .00217. Retaining only four places we write this .0022. Hence

$$\tan 15° 27' = .2742 + .0022 = .2764.$$

The difference between two successive values in the table is called the *tabular difference* (.0031 above). The proportional part of the tabular difference which is used is called the *correction* (.0022 above), and is found by multiplying the tabular difference by the appropriate fraction (.7 above).

Example 1. Find sin 63° 52′.

We find

$$\sin 63° 50' = .8975.$$

tabular difference = .0013 (subtracted mentally from the table).

correction = .2 × .0013 = .0003 (to be added).

Hence,

$$\sin 63° 52' = .8978.$$

* The use of Table I. is explained on pages 80–86 of the text.

Example 2. Find tan 37° 44′.

$$\tan 37° 40' = .7720$$
$$\text{tabular difference} = .0046$$
$$\text{correction} = .4 \times .0046 = .0018.$$

Hence,

$$\tan 37° 44' = .7738.$$

Example 3. Find cos 65° 24′.

$$\cos 65° 20' = .4173$$
$$\text{tabular difference} = 26; \qquad .4 \times 26 = 10$$

(to be subtracted because the cosine decreases as the angle increases). Hence

$$\cos 65° 24' = .4163.$$

Example 4. Find ctn 32° 18′.

$$\text{ctn } 32° 10' = 1.5900$$

tabular difference = 102; $.8 \times 102 = 82$ (to be subtracted). Hence,

$$\text{ctn } 32° 18' = 1.5818.$$

Rule. *To find a trigonometric function of an angle by interpolation: select the angle in the table which is next smaller than the given angle, and read its sine (cosine, tangent, or cotangent as the case may be) and the tabular difference. Compute the correction as the proper proportional part of the tabular difference. In case of sines or tangents* ADD *the correction: in case of cosines or cotangents,* SUBTRACT *it.*

2. Reverse Readings. Interpolation is also used in finding the angle when one of its functions is given.

Example 1. Given $\sin x = .3294$, to find x.

Looking in the table we find the sine which is next less than the given sine to be .3283, and this belongs to 19° 10′. Subtract the value of the sine selected from the given sine to obtain the actual difference = .0011; note that the tabular difference = .0028. We may state the problem as follows: an increase of .0028 in the function increases the angle 10′; then an increase of .0011 in the function will increase the angle 11/28 of 10 = 4 (to be added). Hence x = 19° 14′.

Example 2. Given $\cos x = .2900$, to find x.

The cosine in the table next less than this is .2896 and belongs to 73° 10′; the tabular difference is 28; the actual difference is 4; correction = 4/28 of 10 = 1 (to be subtracted). Hence x = 73° 9′.

Rule. *To find an angle when one of its trigonometric functions is given: select from the table the same named function which is next less than the given function, noting the corresponding angle and the tabular difference: compute the actual difference (between the selected value of the function and the given value), divide it by the tabular difference, and multiply the result by* 10; *this gives the correction which is to be added if the given function is sine or tangent, and to be subtracted if the given function is cosine or cotangent.*

3. The logarithms of the trigonometric functions. If it is required to find log sin 63° 52′, the most obvious way is to find sin 63° 52′ = .8978, and then to find in Table I, log .8978 = 9.9532 − 10, but this involves consulting two tables. To avoid the necessity of doing this, Table II gives the logarithms of the sines, cosines, tangents, and cotangents. The student should note that the sines and cosines of all acute angles, the tangents of all acute angles less than 45° and the cotangents of all acute angles greater than 45° are proper fractions, and their logarithms end with − 10, which is not printed in the table, but which should be written down whenever such a logarithm is used.

Example 1. Find log sin 58° 24′.

In the row having 58° 20′ on the right and in the column having sine at the bottom find log sin 58° 20′ = 9.9300 − 10; the tabular difference is 8; correction = .4 × 8 = 3 (to be added). Hence

$$\log \sin 58° 24' = 9.9303 - 10.$$

(In case of sine and tangent *add* the correction.)

Example 2. Find log cos 48° 38′.

log cos 48° 30′ = 9.8213 − 10, tabular difference 15;
.8 × 15 = 12 (subtract) therefore log cos 48° 38′ = 9.8201 − 10.

(In case of cosine and cotangent, subtract the correction.)

Example 3. Given log tan x = 0.0263, to find x.

The log tan in Table II next less than the given one is 0.0253 and belongs to 46° 40′; actual difference is 10; tabular difference is 25; correction = 10/25 of 10 = 4. Hence x = 46° 44′.

Example 4. Given log cos x = 9.9726 − 10, to find x.

The logarithmic cosine next less than the given one is 9.9725 − 10 and belongs to 20° 10′; actual difference = 1; tabular difference = 5; correction = 1/5 × 10 = 2 (subtract). Hence x = 20° 8′.

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Solid Geometry, cloth, 12mo, 106 pp., $ .80

STRONG POINTS

I. The authors and the editor are well qualified by training and experience to prepare a textbook on Geometry.

II. As treated in this book, geometry functions in the thought of the pupil. It means something because its practical applications are shown.

III. The logical as well as the practical side of the subject is emphasized.

IV. The arrangement of material is pedagogical.

V. Basal theorems are printed in black-face type.

VI. The book conforms to the recommendations of the National Committee on the Teaching of Geometry.

VII. Typography and binding are excellent. The latter is the reënforced tape binding that is characteristic of Macmillan textbooks.

"Geometry is likely to remain primarily a cultural, rather than an information subject," say the authors in the preface. "But the intimate connection of geometry with human activities is evident upon every hand, and constitutes fully as much an integral part of the subject as does its older logical and scholastic aspect." This connection with human activities, this application of geometry to real human needs, is emphasized in a great variety of problems and constructions, so that theory and application are inseparably connected throughout the book.

These illustrations and the many others contained in the book will be seen to cover a *wider range* than is usual, even in books that emphasize practical applications to a questionable extent. This results in a better appreciation of the significance of the subject on the part of the student, in that he gains a truer conception of the wide scope of its application.

The logical as well as the practical side of the subject is emphasized.

Definitions, arrangement, and method of treatment are logical. The definitions are particularly simple, clear, and accurate. The traditional manner of presentation in a logical system is preserved, with due regard for practical applications. Proofs, both formal and informal, are strictly logical.

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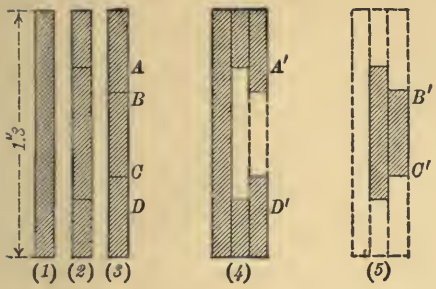
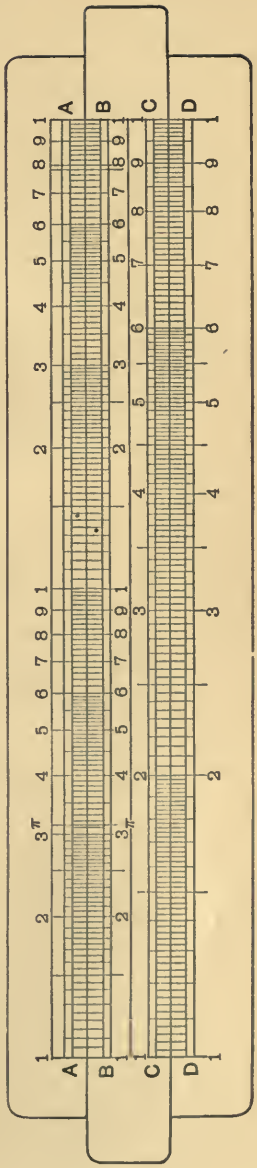


Directions

A reasonably accurate slide-rule may be made by the student, for temporary practice, as follows. Take three strips of heavy stiff cardboard 1″.3 wide by 6″ long; these are shown in cross-section in (1), (2), (3) above. On (3) paste or glue the adjoining cut of the slide rule. Then cut strips (2) and (3) accurately along the lines marked. Paste or glue the pieces together as shown in (4) and (5). Then (5) forms the slide of the slide-rule, and it will fit in the groove in (4) if the work has been carefully done. Trim off the ends as shown in the large cut.



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